Seminar II: Mean-field sparse optimal control

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Outline

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   - Mean-field optimal control
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- **Seminar I:**

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i = 1, 2, \cdots, N, \\ \dot{v}_i = H * \mu_N(x_i, v_i) + f(t, x_i, v_i), \end{cases}$$

where $(x_i, v_i) \in \mathbb{R}^{2d}$. H is some social force, such that, attraction, repulsion, noise, $\cdots$

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where $(x_i, v_i) \in \mathbb{R}^{2d}$. H is some social force, such that, attraction, repulsion, noise, $\cdots$

- **Seminar II:**

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i = 1, 2, \cdots, N, \\ \dot{v}_i = H * \mu_N(x_i, v_i). \end{cases} \tag{1}$$

- Difference: Sparse optimal control.

We rewrite the above system:

$$\begin{cases} \dot{y}_k = w_k, \quad t \in [0, T], \\ \dot{w}_k = H * (\mu_N + \mu_m)(y_k, w_k) + control, \quad k = 1, 2, \cdots, m, \\ \dot{x}_i = v_i, \\ \dot{v}_i = H * (\mu_N + \mu_m)(x_i, v_i), \quad i = 1, 2, \cdots, N, \end{cases} \tag{2}$$

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- Examples in true life:
  - shepherd dog strategy: How many dogs we need?
  - Social system: Can a government endowed with limited resources / rescue / stabilize a society by minimal intervention?

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In the following, we will study:

- **Why we need to study sparse optimal control?**
- **How?**

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Optimal control system 𝟙

We consider the Cucker-Smale model

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i = 1, 2, \cdots, N, \\ \dot{v}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} a(\|x_j - x_i\|)(v_j - v_i) + u_i, \end{cases}$$

where $a(r) = \frac{1}{(1+r^2)^\beta}$, $\beta > 0$ is the communication weight.

- Cost function:

$$\min\{B(u,v) + \gamma(B(x,x))\frac{1}{N}\sum_{i=1}^{N}\|u_i\|\} \quad \text{subject to} \quad \sum_{i=1}^{N}\|u_i\| \leq M,$$

Optimal control system $\mathbb{1}$

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Optimal control system 1

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where

$$B(u,v) := \frac{1}{2N^2}\sum_{i,j}\langle u_i - u_j, v_i - v_j\rangle$$
$$= \frac{1}{N}\sum_{i=1}^{N}\langle u_i - \bar{u}, v_i - \bar{v}\rangle.$$

And

$$\gamma(X) = \int_{\sqrt{X}}^{\infty} a(\sqrt{2N}s)ds.$$

- Reference: [Caponigro-F-Piccoli-Trélat]

Theorem 1.1 [Ha-Liu]

Let $(x, v) \in \mathcal{C}^1([0, \infty), \mathbb{R}^{2d \times N})$ be a solution to Cucker-Smale system. Then

(i) If $\beta \leq \frac{1}{2}$, the solution tends to consensus for every initial data.

(ii) If $\beta > \frac{1}{2}$, let (x^0, v^0) satisfy

$$\sqrt{V^0} < \int_{\sqrt{X^0}}^{\infty} a(\sqrt{2N}r)dr, \tag{3}$$

where

$$X(t) = \frac{1}{N}\sum_{j=1}^{N} \|x_j - \bar{x}\|^2, \quad V(t) = \frac{1}{N}\sum_{j=1}^{N} \|v_j - \bar{v}\|^2.$$

Then, the solution with initial data (x^0, v^0) tends to consensus.

Definition 1.2 (Consensus region)

We call the set of $(x, v) \in \mathbb{R}^N \times \mathbb{R}^N$ satisfying (3) in Theorem 1 the consensus region.

Remark A.1

$$X(t) = B(x, x), \quad V(t) = B(v, v).$$

Aim: to find admissible controls steering the system to consensus region in finite time.

- Set

$$\mathcal{V}_f = \{v \in (\mathbb{R}^d)^N \big| v_1 = \cdots = v_N\},$$

$$\mathcal{V}_\perp = \{v \in (\mathbb{R}^d)^N \big| \sum_{i=1}^N v_i = 0\}.$$

$\Longrightarrow$

(i) The whole space can be decomposed as

$$(\mathbb{R}^d)^N = \mathcal{V}_f \oplus \mathcal{V}_\perp.$$

(ii)

$$V(t) = B(v, v) = \frac{1}{N} \sum_{i=1}^N \|v_{\perp i}\|^2.$$

Definition 1.3

We select the sparse feedback control $u^0 = u^0(x, v) \in U_{ad}(x, v)$ (the set of admissible control) according to the following criterion:

(i) If $\max_{1 \le i \le N} \|v_{\perp i}\| \le \gamma(B(x, x))$, then $u^0 = 0$,

(ii) If $\max_{1 \le i \le N} \|v_{\perp i}\| > \gamma(B(x, x))$, let $j \in \{1, \cdots, N\}$ be the smallest index such that

$$\|v_{\perp j}\| = \max_{1 \le i \le N} \|v_{\perp i}\|.$$

then

$$u_j^0 = -M \frac{v_{\perp j}}{\|v_{\perp j}\|}, \quad u_i^0 = 0 \quad \text{for every} \quad i \neq j. \tag{4}$$

Theorem 1.4

For every $M > 0$, there exists a solution of the optimal control system $\mathbb{1}$ associated with the sparse control u^0 as in Definition 1.4. Fix $M > 0$ and consider the control u^0 law given by Definition 1.3. Then for every initial condition $(x^0, v^0) \in \mathbb{R}^N \times \mathbb{R}^N$, there exists $\tau_0 > 0$ small enough, such that for all $\tau \in (0, \tau_0]$ the sampling solution of optimal control system $\mathbb{1}$ associated with the control u^0, the sampling time τ, and initial pair (x^0, v^0) reaches the consensus region in finite time.

- The choice of u^0 is the best!

We can refer to [Bongini-F-Fröhlich-Haghverdi], [Bongini-F-Kalise] for other results about the general Cucker-Smale model.

[F-Solombrino]

A natural question: What if $N \to \infty$? "Curse of dimensionality"

Particle Cucker-Smale model $\Rightarrow$ $u \to \delta$, as $N \to \infty$.

Thus, we consider **(Smooth controls)**

$$\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H * \mu + f)\mu],$$

with f has some good property. **(Xiongtao)**

Question: What is f_∞? It is an open problem.

Thus, we need to study: **(Sparse optimal control)**

For the other results, for example, use Johnson - Lindenstrauss matrix to lower dimension, PMP, $\cdots$, the history is written by you!

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We add controls on the m leaders to system (7)

$$\begin{cases} \dot{y}_k = w_k, \quad t \in [0, T], \\ \dot{w}_k = H * (\mu_N + \mu_m)(y_k, w_k) + u_k, \quad k = 1, 2, \cdots, m, \\ \dot{x}_i = v_i, \\ \dot{v}_i = H * (\mu_N + \mu_m)(x_i, v_i), \quad i = 1, 2, \cdots, N, \end{cases} \tag{5}$$

where H is a locally Lipschitz interaction kernel with sublinear growth, $u = (u_1, \cdots, u_m) \in L^1([0, T]; U_{ad})$ with U_{ad} a fixed non-empty compact subset of $\mathbb{R}^{d \times m}$ and $U_{ad} \subseteq B(0, R)$ for $R > 0$.

For given initial datum

$$(y_1(0), \cdots, y_m(0), w_1(0), \cdots, w_m(0), x_1(0), \cdots, x_N(0), v_1(0), \cdots, v_N(0)) \in (\mathbb{R}^d)^m \times (\mathbb{R}^d)^m \times (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N,$$

We consider the following optimal control problem:

$$\min_{u \in L^1([0,T];U_{ad})} \int_0^T \{L(y(t), w(t), \mu_N(t)) + \frac{1}{m} \sum_{k=1}^{m} |u_k|\} dt, \tag{6}$$

where

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N}\delta(x_i, v_i), \quad \mu_m = \frac{1}{m}\sum_{k=1}^{m}\delta(y_k, w_k),$$

are the time-dependent atomic measures supported on the phase space trajectories $(y_k(t), w_k(t)) \in \mathbb{R}^{2d}$, for $k = 1, \cdots, m$ and $(x_i(t), v_i(t)) \in \mathbb{R}^{2d}$, for $i = 1, \cdots, N$, respectively, constrained by being the solution of the system (9) with initial data $\zeta_0 = (y^0, w^0, x^0, v^0)$

Remark A.2

(i) We use the 1- Wasserstein distance for μ.

(ii) If

$$\mu_m = \frac{1}{m}\sum_{k=1}^{m}\delta_{\xi_k}, \quad \mu'_m = \frac{1}{m}\sum_{k=1}^{m}\delta_{\xi'_k}.$$

are two atomic measures, then we immediately have that

$$\mathcal{W}_1(\mu_m, \mu'_m) \leq \frac{1}{m}\sum_{k=1}^{m}|\xi_k - \xi'_k|.$$

Assumptions:

(H) Let $H : \mathbb{R}^{2d} \to \mathbb{R}^d$ be a locally Lipschitz function such that, for constant $C > 0$

$$|H(\xi)| \leq C(1 + |\xi|), \quad \text{for all} \quad \xi \in \mathbb{R}^{2d}.$$

(L) Let $L : \mathcal{X} := \mathbb{R}^{2d \times m} \times \mathcal{P}_1(\mathbb{R}^{2d} \to \mathbb{R}_+$ be a continuous function with respect to the distance induced on $\mathcal{X}$ by the norm $\| \cdot \|_{\mathcal{X}}$.

Dynamical system I

Firstly, we consider the existence and uniqueness of solutions for system:

$$\begin{cases} \dot{y}_k = w_k, \quad t \in [0, T], \\ \dot{w}_k = H * (\mu_N + \mu_m)(y_k, w_k) + u_k, \quad k = 1, 2, \cdots, m, \\ \dot{x}_i = v_i, \\ \dot{v}_i = H * (\mu_N + \mu_m)(x_i, v_i), \quad i = 1, 2, \cdots, N, \end{cases} \tag{7}$$

with $u \in L^1([0, T]; U_{ad})$, $U_{ad} \subseteq B(0, R)$, $R > 0$.

Denote $\zeta(t) = (y, w, x, v)$ and $\xi(t) = (y, w)$ or $\xi(t) = (x, v)$, then the system (1.8) can be written

$$\dot{\zeta}(t) = g(t, \zeta(t)),$$

where

$$g(t, \zeta(t)) = \big(w, H*(\mu_N + \mu_m)(y_k, w_k)_{k=1}^m + u_k, v, H*(\mu_N + \mu_m)(x_i, v_i)_{i=1}^N\big).$$

Lemma 2.1

Given H satisfying condition (H) and $\mu_n = \frac{1}{n}\sum_{l=1}^{n}\delta_{\xi_l}$ for $\xi_l \in \mathbb{R}^{2d}$ for all $l = 1, ..., n$, an arbitrary atomic measure, we have

$$|H * \mu_n(\xi)| \leq C\big(1 + \xi + \frac{1}{n}\sum_{l=1}^{n}|\xi_l|\big).$$

Proof.

By sublinear growth of H, we have immediately the estimate

$$|H * \mu_n(\xi)| \leq \frac{1}{n}\sum_{l=1}^{n}|H(\xi - \xi_l)| \leq C\big(1 + \xi + \frac{1}{n}\sum_{l=1}^{n}|\xi_l|\big).$$

□

The existence and uniqueness of solution

Proposition 2.2

Let H be a map satisfying (H). Then, given a control $u \in L^1([0,T], U_{ad})$ and an initial datum $\zeta_0 = (y^0, w^0, x^0, v^0)$, there exists a unique Carathéodory solution $\zeta(t) = (y(t), w(t), x(t), v(t))$ of (9) such that

$$\|\zeta(t)\| \leq (\|\zeta_0\| + \bar{C}T)e^{\bar{C}T},$$

for all $t \in [0,T]$, where $\bar{C} > 0$ is a constant depending on $C > 0$, $R > 0$ but not depending on N. Moreover, the trajectory is Lipschitz continuous in time, i.e.

$$\|\zeta(t_1) - \zeta(t_2)\| \leq \mathcal{L}|t_1 - t_2|, \quad t_1, t_2 \in [0,T],$$

for the Lipschitz constant $\mathcal{L} = \bar{C}(1 + (\|\zeta_0\| + \bar{C}T)e^{\bar{C}T})$.

Proof.

We use the condition (H) and Lemma 3.1 to get

$$\|g(t,\zeta)\| \leq \bar{C}(1+\|\zeta\|), \quad \text{for all} \quad \zeta \in \mathbb{R}^{2d}. \tag{8}$$

By the standard theory about solutions to Carathéodory differential equations (Theorem A.2), the existence and the uniqueness of solutions follows. We use Gronwall's inequality yields that

$$\|\zeta(t)\| \leq (\|\zeta_0\| + \bar{C}T)e^{\bar{C}T}.$$

Moreover, we use (13) to obtain

$$\begin{aligned}\|\zeta(t_1)-\zeta(t_2)\| &= \|\int_{t_1}^{t_2} g(s,\zeta(s))ds\| \leq \int_{t_1}^{t_2} \bar{C}(1+\|\zeta(s)\|)ds \\ &\leq \bar{C}(1+(\|\zeta_0\|+\bar{C}T)e^{\bar{C}T})|t_1-t_2|.\end{aligned}$$

□

Optimal control system II

Now we consider the optimal control system:

$$\min_{u\in L^1([0,T];U_{ad})} \int_0^T \{L(y(t), w(t), \mu_N(t)) + \frac{1}{m}\sum_{k=1}^{m} |u_k|\}dt, \tag{9}$$

with $\zeta(t) := (y(t), w(t), x(t), v(t))$ the solution of the following system:

$$\begin{cases} \dot{y}_k = w_k, \quad t \in [0, T], \\ \dot{w}_k = H * (\mu_N + \mu_m)(y_k, w_k) + u_k, \quad k = 1, 2, \cdots, m, \\ \dot{x}_i = v_i, \\ \dot{v}_i = H * (\mu_N + \mu_m)(x_i, v_i), \quad i = 1, 2, \cdots, N, \end{cases} \tag{10}$$

Theorem 4.1

Under the assumption (H) and (L), the finite horizon optimal control problem II with initial datum $\zeta_0 = (y^0, w^0, x^0, v^0) \in \mathbb{R}^{4d}$ has solutions.

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The coupled ODE and PDE system

The dynamics reads as ($N \to \infty$)

$$\begin{cases} \dot{y}_k = w_k, \quad t > 0, \\ \dot{w}_k = H * (\mu + \mu_m)(y_k, w_k) + u_k, \quad k = 1, 2, \cdots, m, \\ \partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [H * (\mu + \mu_m)\mu], \end{cases} \tag{11}$$

where

$$\mu_m(y_k, w_k) = \frac{1}{m} \sum_{k=1}^{m} \delta(y_k, w_k).$$

The coupled ODE and PDE system

The dynamics reads as ($N \to \infty$)

$$\begin{cases} \dot{y}_k = w_k, \quad t > 0, \\ \dot{w}_k = H * (\mu + \mu_m)(y_k, w_k) + u_k, \quad k = 1, 2, \cdots, m, \\ \partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [H * (\mu + \mu_m)\mu], \end{cases} \tag{11}$$

where

$$\mu_m(y_k, w_k) = \frac{1}{m} \sum_{k=1}^{m} \delta(y_k, w_k).$$

For given initial data (y^0, w^0, μ^0), we consider the following optimal control problem:

$$\min_{u \in L^1([0,T];U_{ad})} \int_0^T \{L(y(t), w(t), \mu(t)) + \frac{1}{m}\sum_{k=1}^{m} |u_k|\} dt.$$

Dynamical system III

Now we consider the existence and stability of solutions to the following dynamical system.

$$\begin{cases} \dot{y}_k = w_k, \quad t > 0, \\ \dot{w}_k = H * (\mu + \mu_m)(y_k, w_k) + u_k, \quad k = 1, 2, \cdots, m, \\ \partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [H * (\mu + \mu_m)\mu], \\ \mu_m(y_k, w_k) = \dfrac{1}{m} \displaystyle\sum_{k=1}^{m} \delta(y_k, w_k), \end{cases} \tag{12}$$

Definition of solution

Definition 3.1

Let $u \in L^1([0,T], U_{ad})$ be given. We say that a map $(y, w, \Delta) : [0,T] \to \mathcal{X} := \mathbb{R}^{2d \times m} \times \mathcal{P}_1(\mathbb{R}^{2d})$ is a solution of the controlled system (11) with control u, if

(i) the measure μ is equi-compactly supported in time, i.e. there exists $R > 0$ such that $\mathrm{supp}(\mu(t)) \in B(0,R)$ for all $t \in [0,T]$;

(ii) the solution is continuous in time with respect to the following metric in $\mathcal{X}$

$$\|(y, w, \mu) - (y', w', \mu')\|_{\mathcal{X}} := \frac{1}{m} \sum_{k=1}^{m} (|y_k - y'_k| + |w_k - w'_k|) + \mathcal{W}_1(\mu, \mu'),$$

where $\mathcal{W}_1(\mu, \mu')$ is the 1-Wasserstein distance in $\mathcal{P}_1(\mathbb{R}^{2d})$;

(iii) the (y, w) coordinates define a Carathéodory solution of the following controlled problem with interaction kernel H, control $u(\cdot)$, and the external field $H * \mu$:

$$\begin{cases} \dot{y}_k = w_k, \quad t > 0, \\ \dot{w}_k = H * (\mu + \mu_m)(y_k, w_k), \quad k = 1, 2, \cdots, m. \end{cases}$$

(iv) the μ component satisfies

$$\frac{d}{dt} \int_{\mathbb{R}^{2d}} \phi d\mu = \int_{\mathbb{R}^{2d}} \nabla \phi \cdot \omega d\mu,$$

for every $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, in the sense of distributions, where $\omega : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$:

$$\omega = (v, H * \mu + H * \mu_m).$$

Let, moreover, $(y^0, w^0, \mu^0) \in \mathcal{X}$ be given, with $\mu^0 \in \mathcal{P}_1(\mathbb{R}^{2d})$ of compact support. We say that $(y, w, \mu) : [0, T] \to \mathcal{X}$ is a solution of (9) with initial data (y^0, w^0, μ^0) and control u if it is a solution of (9) with control u and it satisfies $(y(0), w(0), \mu(0)) = (y^0, w^0, \mu^0)$.

Remark A.3

Similar with [F- Solombrino], once μ_m is a fixed, a measure μ is a weak equi-compactly supported solution of

$$\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [H * (\mu + \mu_m)\mu].$$

in the sense of (iv) in the above definition if and only if it satisfies (i) and the measure - theoretical fixed point equation

$$\mu(t) = \Phi_t^{\mu,\mu_m} \mu^0,$$

where Φ_t^{μ,μ_m} denotes push-forward of μ^0 to $\mu(t)$.

The stability

Proposition 3.2

Let $u \in L^1([0,T], U_{ad})$ be a given fixed control for (9) and two solutions (y^1, w^1, μ^1) and (y^2, w^2, μ^2) of (9) relative to the control u and given respective initial data $(y^{i,0}, w^{i,0}, \mu^{i,0}) \in \mathcal{X}$, with $\mu^{0,i}$, compactly supported, $i = 1, 2$. Then, there exists a constant $C_T > 0$ such that

$$\|(y^1, w^1, \mu^1) - (y^2, w^2, \mu^2)\|_{\mathcal{X}} \leq C_T \|(y^{1,0}, w^{1,0}, \mu^{1,0}) - (y^{2,0}, w^{2,0}, \mu^{2,0})\|_{\mathcal{X}},$$

for all $t \in [0, T]$.

Existence of solution

Theorem 3.3

Let $(y^0, w^0, \mu^0) \in \mathcal{X}$ be given, with μ^0 of bounded support in $B(0, R)$, for $R > 0$. Define a sequence $(\mu_N^0)_{N\in\mathbb{N}}$ of atomic probability measures equi-compactly supported in $B(0, R)$ such that each μ_N^0 is given by $\mu_N^0 = \sum_{i=1}^{N} \delta_{x_{i,N}^0, v_{i,N}^0}$ and $lim_{N\to\infty}\mathcal{W}_1(\mu_N^0, \mu^0) = 0$. Fix now a weakly convergent sequence $(u_N)_{N\in\mathbb{N}} \in L^1([0, T]; U_{ad})$ of control functions, i.e. $u_N \rightharpoonup u_*$ in $L^1([0, T]; U_{ad})$. For each initial datum $\zeta_0 = (y^0, w^0, \mu^0)$ depending on N, denote with $\zeta_N(t) = (y_N(t), w_N(t), \mu_N(t)) := (y_N(t), w_N(t), x_N(t), v_N(t))$, the unique solution of the finite-dimensional control problem (1.9) with control u_N. Then, the sequence $(y_N(t), w_N(t), \mu_N(t))$ converges in $\mathcal{C}_0([0, T], \mathcal{X})$ to some (y_*, w_*, μ_*), which is a solution of dynamical system III with initial data (y^0, w^0, μ^0) and control u_*, in the sense of definition 3.1.

Remark A.4

In view of the uniqueness of the solution of the dynamical system III, we do not need to restrict ourselves to a subsequence, but we can infer the convergence of the entire sequence ζ_N to the solution of dynamical system III.

Optimal control system IV

The dynamics reads as

$$\begin{cases} \dot{y}_k = w_k, \quad t > 0, \\ \dot{w}_k = H * (\mu + \mu_m)(y_k, w_k) + u_k, \quad k = 1, 2, \cdots, m, \\ \partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [H * (\mu + \mu_m)\mu], \\ \mu_m(y_k, w_k) = \dfrac{1}{m} \displaystyle\sum_{k=1}^{m} \delta(y_k, w_k), \end{cases} \tag{13}$$

- The cost function:

$$\min_{u \in L^1([0,T]; U_{ad})} \int_0^T \{L(y(t), w(t), \mu(t)) + \frac{1}{m} \sum_{k=1}^{m} |u_k|\} dt, \tag{14}$$

The Γ-limit to the infinite-dimensional optimal control

Definition 5.1 (Γ-convergence)

Let X be a metrizable separable space and $F_N : X \to (-\infty, \infty]$, $N \in \mathbb{N}$ be a sequence of functionals. Then, we say that F_N Γ-converges to F, written as $F_N \xrightarrow{\Gamma} F$, for an $F : X \to (-\infty, \infty]$, if

(i) lim inf-condition: for every $u \in X$ and every sequence $u_N \to u$,

$$F(u) \leq \liminf_{N\to\infty} F_N(u_N);$$

(ii) lim sup-condition: for every $u \in X$ there exists a sequence $u_N \to u$, called recovery sequence, such that

$$F(u) \geq \limsup_{N\to\infty} F_N(u_N).$$

- Fix now an initial datum $(y^0, w^0, \mu^0) \in \mathcal{X}$, with μ^0 compactly supported, $\mathrm{supp}(\mu^0) \subseteq B(0, R)$, $R > 0$.

- Choose a sequence of equi-compactly supported atomic measures μ_N^0, $\mathrm{supp}(\mu_N^0) \subseteq B(0, R)$, $\mu_N^0 = \frac{1}{N}\sum_{i=1}^N \delta_{x_i^0, v_i^0}$ such that

$$\mathcal{W}_1(\mu_N^0, \mu^0) \to 0, \quad N \to +\infty.$$

- We define

$$F_N(u) = \int_0^T \{L(y(t), w(t), \mu_N(t)) + \frac{1}{m}\sum_{k=1}^{m} |u_k|\}dt, \tag{15}$$

where (y, w, x, v) is the solution of

$$\begin{cases} \dot{y}_k = w_k, \\ \dot{w}_k = H * (\mu_N + \mu_m)(y_k, w_k), \quad k = 1, 2, \cdots, m, \\ \dot{x}_i = v_i, \\ \dot{v}_i = H * (\mu_N + \mu_m)(x_i, v_i), \quad i = 1, 2, \cdots, N, \end{cases} \tag{16}$$

with initial data (y^0, w^0, x_N^0, v_N^0).

Similarly, we define

$$F(u) = \int_0^T \{L(y(t), w(t), \mu(t)) + \frac{1}{m}\sum_{k=1}^{m} |u_k|\} dt. \tag{17}$$

Theorem 5.3

Let H and L be maps satisfying conditions (H) and (L) respectively. Given an initial datum $(y^0, w^0, \mu^0) \in \mathcal{X}$ and an approximating sequence μ_N^0, with μ^0, μ_N^0 equi-compactly supported, i.e. $\text{supp}(\mu^0) \bigcup \text{supp}(\mu_N^0) \subseteq B(0, R)$, $R > 0$, for all $N \in \mathbb{N}$, then the sequence of functionals $(F_N)_{N\in\mathbb{N}}$ on $L^1([0, T], U_{ad})$ defined above Γ-converges to the cost functional $F(u)$.

Corollary 5.4

Let H and L be maps satisfying conditions (H) and (L) respectively. Given an initial datum $(y^0, w^0, \mu^0) \in \mathcal{X}$, with μ^0 compactly supported, $\text{supp}(\mu_N^0) \subseteq B(0, R)$, $R > 0$, the optimal control problem

$$\min_{u \in L^1([0,T];U_{ad})} \int_0^T \{L(y, w, \mu) + \frac{1}{m} \sum_{k=1}^{m} |u_k|\} dt,$$

has solutions, where the triplet (y, w, μ) defines the unique solution of the dynamical system III with initial datum (y^0, w^0, μ^0) and control u in the sense of definition 3.1.

Moreover, solutions to the optimal control system IV can be constructed as weak limits u^* of sequences of optimal controls u_N^* of the finite-dimensional problems

$$\min_{u \in L^1([0,T];U_{ad})} \int_0^T \{L(y(t), w(t), \mu_N(t)) + \frac{1}{m}\sum_{k=1}^{m} |u_k|\} dt,$$

where μ_N and $\mu_{m,N}$ are the time-dependent atomic measures supported on the trajectories defining the solution of the dynamical system I with initial datum $(y^0, w^0, x_N^0, v_N^0) \in \mathcal{X}$ and control u, and $\mu_N^0 = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^0, v_i^0}$ such that

$$\mathcal{W}_1(\mu_N^0, \mu^0) \to 0, \quad N \to +\infty.$$

Thank you!