Model Neurons I: Neuroelectronics

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1. Electrical Properties of Neurons
    - The structure of A Cell
    - Physical Properties of Membrane: Capacitor and Resistance

2. Neuron Models
    - Single-Compartment Model
    - Integrate-and-Fire Model
        - Interspike-interval firing rate
        - Spike-Rate Adaptation
        - Voltage-Dependent Conductances

A cell is composed of nucleus, cytoplasm and **membrane**, etc.

There are microscopic particles (e.g. **ions** and molecules) inside and outside the cell.

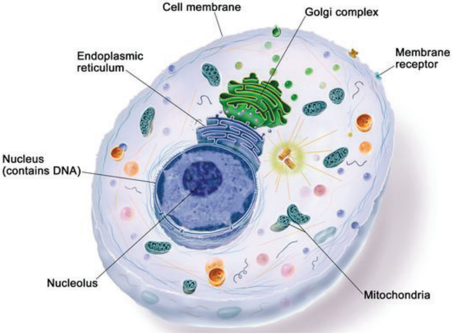

Figure 1: The structure of the animal cell.

The cell membrane is a lipid bilayer 3 to 4 nm thick.

- Insulating feature: impermeable to most charged molecules (C_m).
- Highly selective (ion) channels (R_m).

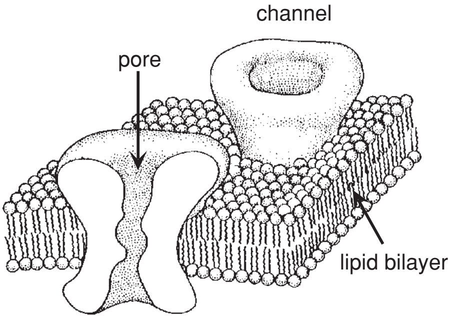

Figure 2: The cell membrane.

Membrane potentials are small enough to allow neurons to take advantage of thermal energy to help transport ions across the membrane, but are large enough so that the thermal fluctuations do not swamp the signaling capabilities of the neurons.

The thermal energy is equal to the electric potential energy, i.e.,

$$RT = V_T F \tag{1}$$

where V_T is the membrane potential, T is the temperature; R is universal gas constant, F is Faraday's constant.

By (1), we have

$$V_T = \frac{RT}{F} \tag{2}$$

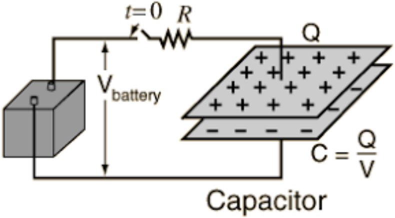

Figure 3: The capacitor: relation between the amount of charge and the voltage.

$$C_m V = Q \tag{3}$$

Taking the time derivative of (3) yields

$$C_m \frac{dV}{dt} = \frac{dQ}{dt} \tag{4}$$

According to the Ohm's law, the voltage deviation can be written by

$$\Delta V = I_{\mathrm{e}} R_{\mathrm{m}}(V) \tag{5}$$

where I_{e} is the injected current (from outside), R_{m} is the resistance, and ΔV is the voltage deviation.

The conductance is defined by the reciprocal of the resistance

$$g_i(V) = \frac{1}{R_{\mathrm{m}}(V)} \tag{6}$$

The channels are heterogeneous.

Negative membrane potentials attract positive ions into the neuron and repel negative ions. In addition, ions diffuse through channels because the ion concentrations differ inside and outside the neuron.

By the Boltzmann distribution, the ratio of the active enough ions who can finish the energy conversion is $e^{\frac{zE}{V_T}}$ under the temperature T. In a balance case, we have

$$[\text{outside}] = [\text{inside}]e^{\frac{zE}{V_T}} \tag{7}$$

$$E = \frac{V_T}{z}\ln\left(\frac{[\text{outside}]}{[\text{inside}]}\right) \tag{8}$$

ions	equilibrium
K^+	-90∼-70 mV
Ca^{2+}	150mV
Na^+	50mV

Table 1: Equilibrium of various ions.

- $V < E$: Ka^+ will go from outside to inside to decrease the membrane potential
- $V > E$: Na^+ and Ca^{2+} will go from outside to inside to decrease the membrane potential.

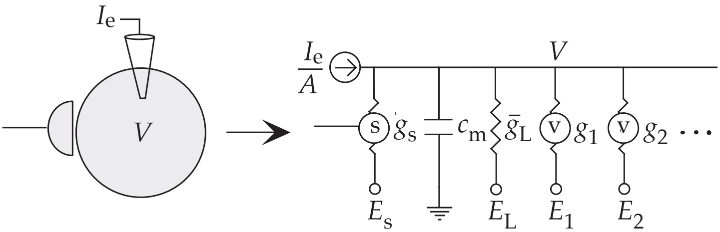

Figure 4: The equivalent circuit for a one-compartment neuron model.

Define $c_{\mathrm{m}} = \frac{C_{\mathrm{m}}}{A}$ be the capacitor in the unit area. By the Kirchhoff law, we have

$$\mathrm{c_m}\frac{dV}{dt} = -i_{\mathrm{m}} + \frac{I_e}{A} \tag{9}$$

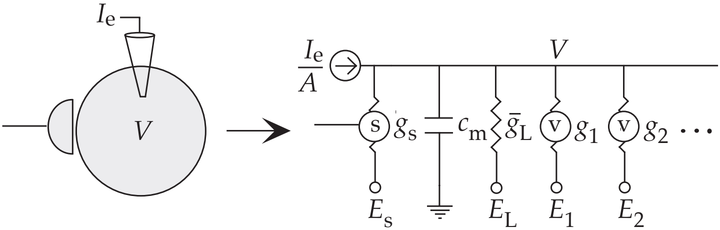

Figure 5: The equivalent circuit for a one-compartment neuron model.

$$i_{\mathrm{m}} = \sum_i g_i (V - E_i) + \bar{g}_{\mathrm{L}} (V - E_{\mathrm{L}}))$$

$$\frac{dV(t)}{dt} = -\frac{1}{c_{\mathrm{m}}} \left(\sum_i g_i + \bar{g}_{\mathrm{L}} \right) V + \frac{1}{c_{\mathrm{m}}} \left(\frac{I_e}{A} - \sum_i g_i E_i - \bar{g}_{\mathrm{L}} E_{\mathrm{L}} \right) \tag{10}$$

The 2nd Model: Integrate-and-Fire Model

A neuron will fire an action potential when $V \in [-55, -50]$. In the following, we model the action potential.

To simplify, we let $g_i = 0$. Then, we have

$$\mathrm{c_m}\frac{dV}{dt} = -\bar{g}_\mathrm{L}(V - E_\mathrm{L}) + \frac{I_\mathrm{e}}{A} \tag{11}$$

Define $r_\mathrm{m} = R_\mathrm{m} \cdot A$ be the unit resistance in the unit area and $\tau_\mathrm{m} = \mathrm{c_m} r_\mathrm{m}$ be a time constant. Multiplying r_m at both sides give

$$\tau_\mathrm{m}\frac{dV}{dt} = E_\mathrm{L} - V + R_\mathrm{m} I_\mathrm{e} \tag{12}$$

The 2nd Model: Integrate-And-Fire Model

$$\tau_\mathrm{m}\frac{dV}{dt} = E_\mathrm{L} - V + R_\mathrm{m}I_\mathrm{e}$$

Solving (13) yileds

$$V(t) = E_\mathrm{L} + R_\mathrm{m}I_\mathrm{e} + (V(0) - E_\mathrm{L} - R_\mathrm{m}I_\mathrm{e})e^{-\frac{t}{\tau_\mathrm{m}}} \tag{13}$$

Let $V(0) = V_\mathrm{reset}$ and t_isi be the time reach thereshold. One can calculate $t = t_\mathrm{isi}$ by

$$V(t_\mathrm{isi}) = V_\mathrm{th} = E_\mathrm{L} + R_\mathrm{m}I_\mathrm{e} + (V_\mathrm{reset} - E_\mathrm{L} - R_\mathrm{m}I_\mathrm{e})e^{-\frac{t_\mathrm{isi}}{\tau_\mathrm{m}}} \tag{14}$$

Solving (14) gives the following **interspike-interval firing rate**

$$r_\mathrm{isi} = \frac{1}{t_\mathrm{isi}} = \begin{cases} \left(\tau_\mathrm{m}\ln\left(\frac{R_\mathrm{m}I_\mathrm{e} + E_\mathrm{L} - V_\mathrm{reset}}{R_\mathrm{m}I_\mathrm{e} + E_\mathrm{L} - V_\mathrm{th}}\right)\right)^{-1}, & R_\mathrm{m}I_\mathrm{e} > V_\mathrm{th}E_\mathrm{L} \\ 0, & R_\mathrm{m}I_\mathrm{e} \leq V_\mathrm{th} - E_\mathrm{L} \end{cases}$$

When I_e is large enough, by the approximation $\ln(x+1) = x$, one has

$$r_{isi} = \left[\frac{R_m}{\tau_m(V_{th} - V_{reset})} I_e + \frac{E_L - V_{th}}{\tau_m(V_{th} - V_{reset})}\right]_+$$

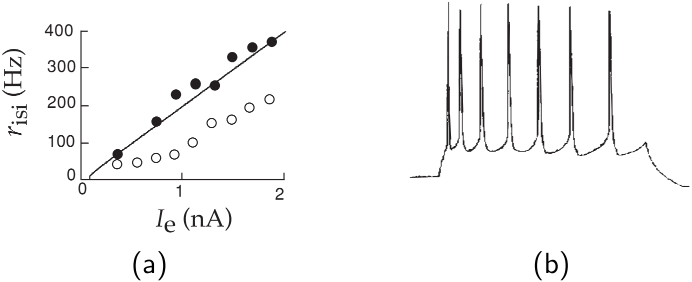

Figure 6: Data fitting performance of the interspike-interval firing rate.

To model the spike-rate adaptation process, we add a term $-r_{\mathrm{m}} g_{\mathrm{sra}}(V - E_{\mathrm{K}})$ in the model

$$\tau_{\mathrm{m}} \frac{dV}{dt} = E_{\mathrm{L}} - V - r_{\mathrm{m}} g_{\mathrm{sra}}(V - E_{\mathrm{K}}) + R_{\mathrm{m}} I_{\mathrm{e}} \tag{15}$$

$$\tau_{\mathrm{sra}} \frac{dg_{\mathrm{sra}}}{dt} = -g_{\mathrm{sra}} \tag{16}$$

where $g_{\mathrm{sra}}(t_k) = (\lim_{t \to t_k^-} g_{\mathrm{sra}}(t)) + \Delta g_{\mathrm{sra}}$.

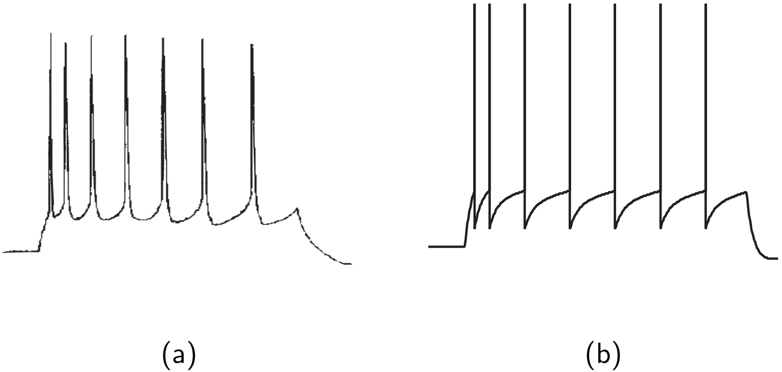

Figure 7: Estimated state trajectories.

Most of the interesting properties arise from nonlinearities associated with active membrane conductances g_i.

g_i depends on several factors

1. **voltage-dependent membrane conductances**
2. transmitter-dependent membrane conductances
3. Ca^{2+}-dependent membrane conductances

Let $g_i(t) = \bar{g}_i P_i(t)$ be the conductance at time t. The problem is that: how to model the dynamics of $g_i(t)$ [or $P_i(t)$]?

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The voltage-dependent membrane conductances $[g_i(t) = g_i(V(t))]$ are divided into delayed-rectifier K^+ conductance and fast Na^+ conductance

Let $n \in [0, 1]$ be the gating or activation variable. We can write

$$P_{\mathrm{K}} = n^k \tag{17}$$

The dynamics of n can be written by

$$\frac{dn}{dt} = \alpha_n(V)(1 - n) - \beta_n(V)n \tag{18}$$

The key elements are the opening and closing rate $\alpha_n(V)$ and $\beta_n(V)$.

The 2nd Model: Integrate-And-Fire Model

$$\frac{dn}{dt} = \alpha_n(V)(1-n) - \beta_n(V)n$$

Let $\tau_n(V)$ and $n_\infty(V)$ be defined by

$$\tau_n(V) = \frac{1}{\alpha_n(V) + \beta_n(V)} \tag{19}$$

$$n_\infty(V) = \frac{\alpha_n(V)}{\alpha_n(V) + \beta_n(V)} \tag{20}$$

We rewrite the dynamics of the n by

$$\tau_n(V)\frac{dn}{dt} = n_\infty(V) - n \tag{21}$$

There are two ways to model the opening and closing rate

1. Mechanism Model:

$$\alpha_n(V) = A_\alpha e^{-\frac{qB_\alpha}{k_\mathrm{B}T}} = A_\alpha e^{-\frac{B_\alpha V}{V_T}},\ B_\alpha < 0 \tag{22}$$

$$\beta_n(V) = A_\beta e^{-\frac{qB_\beta}{k_\mathrm{B}T}} = A_\beta e^{-\frac{B_\beta V}{V_T}} \tag{23}$$

2. Data Fitting Model:

$$\alpha_n(V) = \frac{0.01(V+55)}{1-e^{-0.1(V+55)}} \tag{24}$$

$$\beta_n(V) = 0.125e^{-0.0125(V+65)} \tag{25}$$

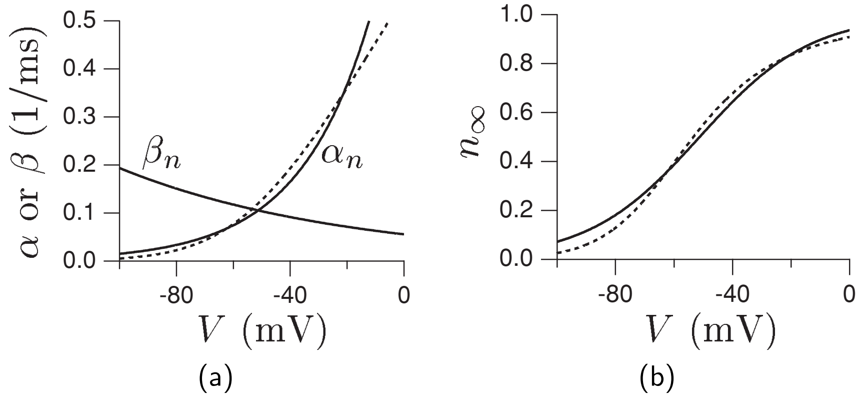

Figure 8: Estimated state trajectories.

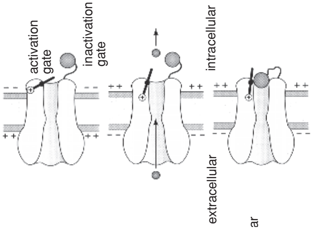

Figure 9: Transient Conductances.

The probability of the opening channels is

$$P_{\mathrm{Na}} = m^k h \tag{26}$$

where m is an activation variable and h is the probability that the ball does not block the cannel pore.

We can do the same things for α_m, β_m, α_h and β_h as $\alpha_n(V)$ and $\beta_n(V)$.

1. Mechanism Model

$$\alpha_m(V) = \bar{A}_\alpha e^{-\frac{q\bar{B}_\alpha}{k_{\mathrm{B}}T}} = \bar{A}_\alpha e^{-\frac{\bar{B}_\alpha V}{V_T}},\ \beta_m(V) = \bar{A}_\beta e^{-\frac{q\bar{B}_\beta}{k_{\mathrm{B}}T}} = \bar{A}_\beta e^{-\frac{\bar{B}_\beta V}{V_T}}$$

$$\alpha_h(V) = \tilde{A}_\alpha e^{-\frac{q\tilde{B}_\alpha}{k_{\mathrm{B}}T}} = \tilde{A}_\alpha e^{-\frac{\tilde{B}_\alpha V}{V_T}},\ \beta_h(V) = \tilde{A}_\beta e^{-\frac{q\tilde{B}_\beta}{k_{\mathrm{B}}T}} = \tilde{A}_\beta e^{-\frac{\tilde{B}_\beta V}{V_T}}$$

2. Data Fitting Model

$$\alpha_m = \frac{0.1(V+40)}{1-e^{-0.1(V+40)}},\ \beta_m = 4e^{-0.0556(V+65)},$$

$$\alpha_h = 0.07e^{-0.05(V+65)},\ \beta_h = \frac{1}{1+e^{-0.1(V+35)}}$$

Conclusion

1. The membrane can be modelled by the capacitor, resistance, and the amount of active (enough) ions can be calculated by Boltzmann distribution.
2. Single-Compartment model is the simplest model by the Kirchhoff's laws.
3. Integrate-and-Fire improves single-compartment in two aspects: spike-rate adaptation, time-varying voltage-dependent conductances.

Thank you for your listening!