Ch 5. Numerical Dynamics

Notations: Throughout the seminar, we will use the following notations.

$\mathbb{T} := \mathbb{R}$ or $\mathbb{Z}$, $\quad \mathbb{T}_0^+ := \mathbb{R}_{\geq 0}$ or $\mathbb{Z}_{\geq 0}$.
(X, d_X) : state space, $\quad (\Omega, \mathcal{F}, \mathbb{P})$: probability space.
P : base space for skew product flow ($P = \Omega$ in our textbook).
$\text{dist}(A, B) := \sup_{x \in A} \inf_{y \in B} d_X(x, y), \quad \forall\ A, B \subset X.$

Two major issues,

1. the preservation of an attractor under discretization
2. a hyperbolic neighbourhood under discretization

are considered here in the context of RODEs and the RDSs. In this seminar, I would like to cover the first topic and omit the second one.

1 Review: Random Dynamical System

Definition 1.1. *A random dynamical system (θ, φ) on $\Omega \times X$ consists of a metric dynamical system $\theta : \mathbb{T} \times \Omega \to \Omega,\ (t, \omega) \mapsto \theta_t(\omega)$, such that*

1. $\theta_0(\omega) = \omega \quad \forall\ \omega \in \Omega,$
2. $\theta_t \circ \theta_s = \theta_{t+s} \quad \forall\ t, s \in \mathbb{T},$
3. *the map* $(t, \omega) \mapsto \theta_t(\omega)$ *is measurable on* $\Omega \times \mathbb{T}$,
4. θ_t *has a measure preserving property, i.e.,*

$$\mathbb{P}(\theta_t(A)) = \mathbb{P}(A) \quad \forall\ t \in \mathbb{T} \text{ and } A \in \mathcal{F},$$

and a cocycle mapping $\varphi : \mathbb{T}_0^+ \times \Omega \times X \to X$ *such that*

1. $\varphi(0, \omega, x_0) = x_0$ *for all* $x_0 \in X$ *and* $\omega \in \Omega$,
2. $\varphi(s+t, \omega, x_0) = \varphi(s, \theta_t(\omega), \varphi(t, \omega, x_0))$ *for all* $x, t \in \mathbb{T}_0^+,\ x_0 \in X$ *and* $\omega \in \Omega$,
3. $(t, x_0) \mapsto \varphi(t, \omega, x_0)$ *is continuous for each* $\omega \in \Omega$,
4. $\omega \mapsto \varphi(t, \omega, x_0)$ *is* $\mathcal{F}$*-measurable for each* $(t, x_0) \in \mathbb{T}_0^+ \times X$.

Note: 1. The cocycle mapping φ can be understood in the following way: let $\pi : \mathbb{T}_0^+ \times \Omega \times X \to \Omega \times X$ be a mapping which is given by

$$\pi_t(\omega, x) = \pi(t, (\omega, x)) := (\theta_t(\omega), \varphi(t, \omega, x)).$$

Then, we have

$$\pi_t \circ \pi_s = \pi_{t+s}, \quad \forall\ t, s \in \mathbb{T}_0^+.$$

2. If Ω is a metric space and the 'measurable' in Definition 1.1 are modified to 'continuous', then (θ, φ) is called a skew product flow, which represents a nonautonomous dynamical system. For instance, if we have a deterministic ODE of the form

$$\dot{p} = f(p), \quad \dot{x} = g(p, x), \quad (p(t_0), x(t_0)) = (p_0, x_0),$$

one can write

$$p(t, t_0, p_0) = \theta_{t-t_0}(p_0), \quad x(t, t_0, p_0, x_0) = \varphi(t - t_0, p_0, x_0).$$

3. Similarly, the RODE

$$\dot{x} = -x + \cos W_t(\omega),$$

where W_t is a two-sided Wiener process and has the explicit solution

$$x(t, t_0, x_0, \omega) = x_0 e^{-(t-t_0)} + e^{-t} \int_{t_0}^{t} e^s \cos W_s(\omega) ds,$$

can be interpreted as RDS, where

$$\begin{aligned}
&\Omega := \{\omega \in C_0(\mathbb{R}, \mathbb{R}) | \omega(0) = 0\}, \\
&\theta_t(\omega)(\cdot) := \omega(t + \cdot) - \omega(t), \\
&\varphi(t, \omega, x_0) := x(t, 0, \omega, x_0).
\end{aligned}$$

(see [Kloeden-Rasmussen 2011])

In addition, it is necessary to recall the following definitions to consider the random attractor:

Definition 1.2. *Let P be a nonempty set and $\phi : \mathbb{T}_0^+ \times P \to P$ be a mapping satisfying the semigroup property, i.e.,*

$$\phi(0, p_0) = p_0, \quad \phi(s + t, p_0) = \phi(s, \phi(t, p_0)), \quad \forall\ s, t \in \mathbb{T}_0^+, \quad \forall\ p_0 \in P. \tag{1}$$

A subset D of P is called ϕ-invariant if

$$\phi(t, D) = D, \quad \forall\ t \in \mathbb{T}_0^+,$$

and called ϕ-positively invariant if

$$\phi(t, D) \subset D, \quad \forall\ t \in \mathbb{T}_0^+.$$

Now, suppose we want to consider a π-invariant set for $\pi = (\theta, \varphi)$. If

$$D = \bigcup_{p \in P} (\{p\} \times D_p), \quad \emptyset \subsetneq D_p \subset X,$$

then D is π-invariant if

$$\varphi(t, p, D_p) = D_{\theta_t(p)}, \quad \forall\, p \in P, \quad t \in \mathbb{T}_0^+,$$

and π-positively invariant if

$$\varphi(t, p, D_p) \subset D_{\theta_t(p)}, \quad \forall\, p \in P, \quad t \in \mathbb{T}_0^+.$$

If (θ, φ) is a RDS, then it is reasonable to require some sort of measurability.

Definition 1.3. *Let $\pi = (\theta, \varphi) : \mathbb{T}_0^+ \times \Omega \times X \to \Omega \times X$ be a RDS and*

$$D = \bigcup_{\omega \in \Omega} \left(\{\omega\} \times D_\omega \right), \quad \emptyset \subsetneq D_\omega \subset \Omega.$$

Then, D is called a random set if it is $\mathcal{X} \times \mathcal{F}$-measurable, where $\mathcal{X}$ is the Borel σ-algebra of X. A random set D is called a random closed set (resp. compact) if each D_ω is closed (resp. compact), and D said to be tempered if $\exists\, x_0 \in X$ such that

$$D_\omega \subset \{x \in X : d(x, x_0) \leq r(\omega)\} \quad \forall\, \omega \in \Omega,$$

where the random variable $r(\omega)$ has a sub-exponential growth:

$$\sup_{t \in \mathbb{R}} \left\{ r(\theta_t(\omega)) e^{-\gamma|t|} \right\} < \infty \quad \forall\, \gamma > 0, \quad \omega \in \Omega.$$

Note: For a(n autonomous) dynamical system $\phi : \mathbb{T}_0^+ \times X \to X$, a set $A \subset X$ is called an attractor if A is a nonempty compact ϕ-invariant subset of X and

$$\lim_{t \to \infty} \operatorname{dist}(\phi(t, B), A) = 0, \quad \forall\, B : \text{bounded}.$$

To define an attractor for $\pi = (\theta, \varphi)$, we may use the fiber-wise distance between attractor $A \subset P \times X$ and set $\pi_t(D)$. For each $p \in P$, the p-fiber of $\pi_t(D)$ is $\varphi(t, \theta_{-t}(p), D_{\theta_{-t}(p)})$, since

$$\pi_t(\theta_{-t}(p), D_{\theta_{-t}(p)}) = (p, \varphi(t, \theta_{-t}(p), D_{\theta_{-t}(p)})).$$

Therefore, for each $p \in P$, we estimate

$$\operatorname{dist}\left(\varphi(t, \theta_{-t}(p), D_{\theta_{-t}(p)}), A_p \right).$$

Then, should A attract all fiber-bounded subsets of $P \times X$? For detailed analysis it is good to relax this requirement.

Definition 1.4. *An attraction universe $\mathcal{D}$ of (θ, φ) is a collection of subset of $P \times X$ with bounded p-fibers where*

$$\emptyset \subsetneq D' \subset D, \quad D \in \mathcal{D} \quad \textit{implies} \quad D' \in \mathcal{D}.$$

Note: If (θ, φ) is a random dynamical system, the collection of all tempered random sets $\mathcal{T}$ is an attraction universe.

Now, we are ready to define the pullback attractor.

Definition 1.5. *For given $\pi := (\theta, \varphi)$, a nonempty, fiber-compact and π-invariant set $A \subset P \times X$ is called pullback attractor with respect to an attraction universe $\mathcal{D}$ if*

$$\lim_{t\to\infty} \text{dist}\left(\varphi(t, \theta_{-t}(p), D_{\theta_{-t}(p)}), A_p\right) = 0, \quad \forall\, p \in P, \quad \forall\, D \in \mathcal{D}.$$

If π is a RDS and $\mathcal{D} = \mathcal{T}$, then A is called a random attractor.

Similarly, the pullback absorbing set can be defined as follows.

Definition 1.6. *For given $\pi = (\theta, \varphi)$ and an attraction universe $\mathcal{D}$, a nonempty, fiber-compact set $K \in \mathcal{D}$ is called pullback absorbing with respect to $\mathcal{D}$ if*

$$\forall\, D \in \mathcal{D}, \quad \forall\, p \in P, \quad \exists\, T = T(p, D) > 0 \quad \textit{such that}$$
$$\varphi(t, \theta_{-t}(p), D_{\theta_{-t}(p)}) \subset K_p \quad \forall\, t \geq T.$$

In general, it is known that the existence of absorbing set implies the existence of attractor. The following theorem shows such a result for skew product flow (θ, φ).

Theorem 1.1. *For given $\pi = (\theta, \varphi)$ and attracting universe $\mathcal{D}$, let K be a pullback absorbing set with respect to $\mathcal{D}$. Then, (θ, φ) has a pullback attractor A with respect to $\mathcal{D}$, where*

$$A_p = \bigcap_{s>0} \overline{\bigcup_{t\geq s} \varphi(t, \theta_{-t}(p), K_{\theta_{-t}(p)})}.$$

In addition, $A_p \subset K_p$ for all $p \in P$.

We refer Chapter 3 of the book [Kloeden-Rasmussen 2011] for readers who are interested in the proof. The proof is essentially the same when (θ, φ) is RDS. The only new thing here is to show that A is a random set. This follows from the fact that

$$\omega \mapsto \varphi(t, \theta_{-t}(\omega), K_{\theta_{-t}(\omega)})$$

is measurable for each $t \in \mathbb{T}_0^+$ and the intersection and union can be taken over a countable number of times(union? without positive invariance?).

2 Discretization of Random Attractors

In Chapter 4, the existence of a random pullback attractor A with respect to $\mathcal{T}$ was established for RDS (θ, φ) generated by the RODE

$$\dot{x} = \mu(x) + \zeta(\theta_t(\omega)), \tag{2}$$

where $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ satisfies

$$(\mathcal{C}1): \quad \langle \mu(x), x \rangle \leq -M\|x\|^2 + N^2$$

for some $M > 0$ and $\zeta : \Omega \to \mathbb{R}^d$ satisfies

$$(\mathcal{C}2): \quad \lim_{|t|\to\infty} e^{-c|t|}\|\zeta(\theta_t(\omega))\| = 0$$

for any $\omega \in \Omega$ and $c > 0$.

The implicit Euler scheme with constant step size $h > 0$ to (2) is given by

$$x_{n+1} = x_n + h\left[\mu(x_{n+1}) + \zeta(\theta_{nh}(\omega))\right] \tag{3}$$

For h sufficiently small to have $\mathrm{Id} - h\mu$ an inverse, (3) can be written as

$$x_{n+1} = G_h(x_n, \theta_{nh}(\omega)), \quad n \in \mathbb{T}_0^+ := \mathbb{Z}_{\geq 0}.$$

Then, $\pi_n(\omega, x_0) = (\theta_{nh}(\omega), \psi_h(n, \omega, x_0)) := (\theta_{nh}(\omega), x_n)$ is a (discrete time) RDS on $\Omega \times \mathbb{R}^d$, and we have the following result.

Theorem 2.1. *Suppose μ and ζ satisfy the conditions $(\mathcal{C}1) - (\mathcal{C}2)$. Then,* (3) *has a random attractor A_h for sufficiently small $h > 0$. Moreover, A_h converges to the random attractor A for RODE* (2) *upper semicontinuously, i.e.,*

$$\lim_{h\to 0} \mathrm{dist}(A_h(\omega), A(\omega)) = 0, \quad \forall\, \omega \in \Omega. \tag{4}$$

Proof. As a consequence of Theorem 1.1, it is sufficient to find a random absorbing set for sufficiently small $h > 0$ to show that (θ, ψ_h) has a random attractor. In particular,

$$\begin{aligned}\|x_{n+1}\|^2 &= \langle x_{n+1}, x_n\rangle + h\langle x_{n+1}, \mu(x_{n+1})\rangle + h\langle x_{n+1}, \zeta(\theta_{nh}(\omega))\rangle \\ &\leq \frac{1}{2}\|x_{n+1}\|^2 + \frac{1}{2}\|x_n\|^2 - hM\|x_{n+1}\|^2 + hN^2 + \frac{1}{2}hM\|x_{n+1}\|^2 + \frac{2h}{M}\|\zeta(\theta_{nh}(\omega))\|^2,\end{aligned}$$

and therefore

$$\|\psi_h(n+1, \omega, x_0)\|^2 \leq \frac{1}{1+hM}\|\psi_h(n, \omega, x_0)\|^2 + \frac{2hN^2}{1+hM} + \frac{4h}{M(1+hM)}\|\zeta(\theta_{nh}(\omega))\|^2.$$

Writing $\lambda := \frac{1}{1+hM}$, we have

$$\begin{aligned}\|\psi_h(n, \omega, x_0)\|^2 &\leq \lambda^n\|\psi_h(0, \omega, x_0)\|^2 + (\lambda + \cdots + \lambda^n)2hN^2 + \sum_{j=1}^{n}\frac{4h\lambda^j}{M}\|\zeta(\theta_{(n-j)h}(\omega)\|^2 \\ &\leq \lambda^n\|x_0\|^2 + \frac{2hN^2\lambda}{1-\lambda} + \frac{4h}{M}\sum_{j=1}^{n}\lambda^j\|\zeta(\theta_{(n-j)h}(\omega)\|^2 \\ &= \lambda^n\|x_0\|^2 + \frac{2N^2}{M} + \frac{4h}{M}\sum_{j=1}^{n}\lambda^j\|\zeta(\theta_{(n-j)h}(\omega)\|^2,\end{aligned}$$

and substituting $\theta_{-nh}(\omega)$ instead of ω, we have

$$\|\psi_h(n, \theta_{-nh}(\omega), x_0)\|^2 \leq \lambda^n\|x_0\|^2 + \frac{2N^2}{M} + \frac{4h}{M}\sum_{j=1}^{n}\lambda^j\|\zeta(\theta_{-jh}(\omega)\|^2.$$

Now, let K_h be a random set where each ω-fiber is a closed ball with center 0 and radii

$$r_h(\omega) := \left(1 + \frac{2N^2}{M} + \frac{4h}{M}\sum_{j=1}^{\infty} \lambda^j \|\zeta(\theta_{-jh}(\omega)\|^2\right)^{\frac{1}{2}}.$$

Then for every $x_0 \in D(\theta_{-nh}(\omega))$ with $D \in \mathcal{T}$, we have

$$\|\psi_h(n, \theta_{-nh}(\omega), x_0)\|^2 \leq \lambda^n \sup_{x \in D(\theta_{-nh}(\omega))} \|x\|^2 + r_h(\omega)^2 - 1 \leq r_h(\omega)^2,$$

for sufficiently large n, since

$$\lim_{n\to\infty} \lambda^n \sup_{x \in D(\theta_{-nh}(\omega))} \|x\|^2 = 0$$

as D is tempered and $\lambda^n \simeq e^{-nhM}$ for sufficiently small h. In addition, the condition $(\mathcal{C}2)$ shows $r_h(\omega)$ is tempered. Therefore, the set

$$K_h = \bigcup_{\omega\in\Omega} (\{\omega\} \times K_h(\omega)), \quad K_h(\omega) := \bar{B}_{r_h(\omega)}(0),$$

is a random absorbing set for (θ, ψ_h). Therefore, by using Theorem 1.1, we have the existence of random attractor A_h for sufficiently small h.

For the upper semicontinuous convergence (4), suppose there exists $\varepsilon_0 > 0$, $\omega \in \Omega$, sequence $\{h_n\}_{n\geq 0}$ with $\lim_{n\to\infty} h_n = 0$ and points $a_n \in A_{h_n}(\omega)$ such that

$$\operatorname{dist}(a_n, A(\omega)) > \varepsilon_0 \tag{5}$$

for all $n \geq 0$. Since A is an attractor of (2) with respect to $\mathcal{T}$, there is a $T_0 > 0$ such that

$$\operatorname{dist}(\varphi(t, \theta_{-t}(\omega), K_h(\theta_{-t}(\omega))), A(\omega)) < \frac{1}{4}\varepsilon_0, \quad \forall\, t \geq T_0.$$

The global discretization error of (3) on an interval $[0, T_0]$ starting at $\xi \in K_h(\theta_{-T_0}(\omega))$ is

$$\|\psi_h(j, \theta_{-T_0}(\omega), \xi) - \varphi(jh, \theta_{-T_0}(\omega), \xi)\| \leq C_{T_0}(\omega)h^q, \quad 0 \leq j \leq \frac{T_0}{h},$$

where q is determined by the Hölder continuity exponent of $\zeta(\theta_t(\omega))$. Define

$$h_0^* = \left[\frac{\varepsilon_0}{4C_{T_0}(\omega)}\right]^{\frac{1}{q}}$$

and pick (h, N_h) such that $h \leq h_0^*$, $N_h h = T_0$. Then

$$\|\psi_h(j, \theta_{-T_0}(\omega), \xi) - \varphi(jh, \theta_{-T_0}(\omega), \xi)\| \leq \frac{\varepsilon_0}{4}, \quad 0 \leq j \leq \frac{T_0}{h}.$$

From the invariance property of the attractor A_{h_n}, one can find

$$\xi_n \in A_{h_n}(\theta_{-T_0}(\omega)) \subset K_h(\theta_{-T_0}(\omega))$$

such that $\psi_{h_n}(N_h, \theta_{-T_0}(\omega), \xi_n) = a_n$. Therefore, for each $n \geq 0$,

$$\begin{aligned}
\operatorname{dist}(a_n, A(\omega)) &= \operatorname{dist}(\psi_{h_n}(N_h, \theta_{-T_0}(\omega), \xi_n), A(\omega)) \\
&\leq \|\psi_{h_n}(N_h, \theta_{-T_0}(\omega), \xi_n) - \varphi(T_0, \theta_{-T_0}(\omega), \xi_n)\| \\
&\quad + \operatorname{dist}(\varphi(T_0, \theta_{-T_0}(\omega), \xi_n), A(\omega)) \\
&< \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} = \frac{\varepsilon_0}{2},
\end{aligned}$$

which contradicts (5). □

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