QFT LECTURE NOTE 13-16

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0.1. Brief review.

- Lec 13
  Let we start from the brief review. Because in the last lecture, we defined several new concepts, we should remind it.
  In lecture 12, we defined the "massive scalar free field" And in lecture 13, we proved that how the massive scalar free field evolves:

  Proposition 0.1. *(Prop 13.1)[Evolution of massive free field] For any* $(t, \mathbf{x}) \in \mathbb{R}^{1,3}$*, the massive free field satisfies*
  $$\varphi(t, \mathbf{x}) = e^{itH_0}\varphi(0, \mathbf{x})e^{-itH_0}$$
  (We mentioned that the interaction Hamiltonian describing the repulsion force can be represented by the "massive scalar free field".:)
  $H = H_0 + gH_I$, (where g is coupling parameter)
  $$H_I = \frac{1}{4!}\int d\mathbf{x} : \varphi(0, \mathbf{x})^4 :$$
- Lec 14
  During lecture 14-15, we considered "Scattering" in quantum mechanics without special relativity and creation or annilhilation of particles. In Lec 16, We will consider scattering in QFT. So we want to remind several definitions in scattering.
  We first note that the free evolution is different in non-quantum case and quantum case. (Draw a table)
  Free evolution, classic, $t = 0$, $(\mathbf{x}, \mathbf{v})$: $(\mathbf{x} + \mathbf{v}t)_{t\in\mathbb{R}}$
  Free evolution, quantum, $t = 0$, $|\psi\rangle$: $(U_0(t)|\psi\rangle)_{t\in\mathbb{R}}$
  (This is called "the trajectory of $|\psi\rangle$.)
  (For $t < 0$, we defined) (Draw a picture)
  $$\Omega_+|\psi\rangle = \lim_{t\to-\infty} U(-t)U_0(t)|\psi\rangle$$
  (For $t > 0$,)
  $$\Omega_-|\psi\rangle = \lim_{t\to\infty} U(-t)U_0(t)|\psi\rangle$$
  where
  $$U(t) = e^{-itH}, \qquad U_0(t) = e^{-itH_0}$$
  • Scattering operator
  $$S := \Omega_-^{-1}\Omega_+$$
  $S|\psi\rangle$: "trajectory of a particle in the far future if it is on the trajectory $|\psi\rangle$ in the far past.
  • Dyson series: (then we obtained a series expansion of the scattering operator.

Note 1: we didn't give any explicit interaction Hamiltonian. Note 2: Without interaction Hamiltonian, scattering operator become just identity operator. If we put a state $|\mathbf{p_1}\rangle$ then the scattering (outgoing particle) state is also the same $|\mathbf{p_1}\rangle$. (We computed it in Lec 15))

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-ig)^n}{n!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} T H_I(\theta_1) H_I(\theta_2) \cdots H_I(\theta_n) d\theta_n \cdots d\theta_1$$

(without interaction, S is just identity operator)

- Lec 15
  (We considered the Born approximation which is first order approximation of Dyson series. We computed two things:)
  (Square of it means the probability density function undergoes the Hamiltonian H when we start from the trjectory of $|\mathbf{p_1}\rangle$).

$$\langle \mathbf{p_2}|S|\mathbf{p_1}\rangle = (2\pi)^3 \delta^3(\mathbf{p_2} - \mathbf{p_1}) + (-ig)(2\pi)\hat{V}(\mathbf{p_2} - \mathbf{p_1})\delta((|\mathbf{p_2}|^{\mathbf{2}} - |\mathbf{p_1}|^{\mathbf{2}})/2m) + O(g^2)$$

(If p_1 is different from p_2 then the probability always zero. So It means that if we put a state p_1 then the outgoing state is also p_1. This is because, we put a improper state Dirac-delta p_1. Thus we try to start from the proper state:)

$$|\psi_\epsilon\rangle = \frac{1}{(2\pi\epsilon)^{3/2}} e^{-(\mathbf{p}-\mathbf{p_1})^2/2\epsilon}$$

$$\langle \mathbf{p_2}|S|\psi_\epsilon\rangle = \underbrace{\frac{C_1}{\epsilon^{3/2}} e^{-(\mathbf{p_2}-\mathbf{p_1})^2/2\epsilon}}_{A(\mathbf{p_2})} + \underbrace{(-ig)C_2(2\pi)\hat{V}(\mathbf{p_2} - \mathbf{p_1})\frac{1}{\sqrt{\epsilon}} e^{-\frac{(|\mathbf{p_2}|^{\mathbf{2}} - |\mathbf{p_1}|^{\mathbf{2}})^{\mathbf{2}}}{8m^2\epsilon}}}_{B(\mathbf{p_2})}$$

(Draw a picture. What is the scattering means?)
Let we compare two pictures. The first case means that if we put the improper state p_1, then the scattering result also is state p_1.
The second case shows that, if we start from some distributed state, then in high probability, it is concentrated near p_1. If the scattering occur, then the probability is proprtional to $(\hat{V}(\mathbf{p_2} - \mathbf{p_1}))^2$.

1. Lecture 16 Hamiltonian densities

1.1. 16.1 Scattering in QFT.

Subject of Lecture 16:

- Def Hamiltonian density $\mathcal{H}$.
- (If the interaction Hamiltonian is constructed from Hamiltonian density,)

$$H_I(0) = \int_{\mathbb{R}^3} d\mathbf{x} \mathcal{H}(0, \mathbf{x})$$

then the Dyson series takes particularly nice form:

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-ig)^n}{n!} \int_{\mathbb{R}^{1,3}} \cdots \int_{\mathbb{R}^{1,3}} dx_1 \cdots dx_n T\mathcal{H}(x_1) \cdots \mathcal{H}(x_n)$$

(dt integral can be expanded to dx^4 integral)

- : φ^4 :(phi four) is Hamiltonian density

$$H_I = \frac{1}{4!}\int_{\mathbb{R}^3} d\mathbf{x} : \varphi(x)^4 :$$

@@@@@@@@@@@@@@@@@@@ (Skip) Recall several definitions in QFT:

$$H_0\psi(p_1, \cdots, p_n) = \left(\sum_{j=1}^{n} p_j^0\right)\psi(p_1, \cdots, p_n)$$

$$a(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}a(p), \qquad a^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}a^\dagger(p)$$

$$\varphi(x) = \int_{\mathbb{R}^3}\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(e^{-i(x,p)}a(\mathbf{p}) + e^{i(x,p)}a^\dagger(\mathbf{p})\right)$$

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In lecture 14-15, Scattering theory, we considered one-particle case. Now we want to extend the theory in relativistic case including the creation and annilhilation of particles. To allow the change of the number of particle, we should consider :

$$\langle \mathbf{q_1}, \cdots, \mathbf{q_j}|S|\mathbf{p_1}, \cdots, \mathbf{p_k}\rangle$$

We will consider it later. Before we apply the momentum state with large number of particle (more than 2), we only focus onthe scattering operator and the Dyson series. (Dyson series is only series for S.)

1.2. **16.2 Construction of interaction Hamiltonians.** (We first define the "Hamiltonian density".)

Definition 1.1. *16.1* A Hamiltonian density is operator-valued distribution satisfying

(1) (Time evolution) $\mathcal{H}(t, \mathbf{x}) = e^{itH_0}\mathcal{H}(0, \mathbf{x})e^{-itH_0}$
(2) (Equal time commutation) For any t, $[\mathcal{H}(t, \mathbf{x}), \mathcal{H}(t, \mathbf{y})] = 0$

(In Lecture 13, we observed that the interaction Hamiltonian about repulsion can be described by the φ^4 interaction Hamiltonian:)

$$H_I = \int_{\mathbb{R}^3} d\mathbf{x}\mathcal{H}(0, \mathbf{x})$$

(In the remaining part, we prove that it is Hamiltonian density.)
We prove $\mathcal{H}(x) = \frac{1}{4!} : \varphi(x)^4 :$ is Hamiltonian density.
We first separate the massive scalar free field:

$$\varphi^-(x) = \int_{\mathbb{R}^3}\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}e^{i(x,p)}a^\dagger(\mathbf{p})$$

$$\varphi^+(x) = \int_{\mathbb{R}^3}\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}e^{-i(x,p)}a(\mathbf{p})$$

Proposition 1.1. 16.1 *$\mathcal{H}(x) =: \varphi(x)^n :$ is Hamiltonian density.*

(We need the following auxiliary lemma.)

Lemma 1.2. 16.1 *For all $n \in \mathbb{N}$, $: \varphi(x)^n :$ is formal polynomial in $\varphi(x)$.*

$: \varphi(x)^n :\neq \varphi(x)^n$
(We first note that $: \varphi(x)^n :$ is generally not equal to $\varphi(x)^n$. But this lemma says that $: \varphi(x)^n :$ can be represented by the polynomial of $\varphi(x)$. The definition of normal ordering is just put all of $a^\dagger$ on the left hand side.)

Proof. We claim that

$$: \varphi(x)^n : \varphi(x) =: \varphi(x)^{n+1} : + Cn : \varphi(x)^{n-1} :$$

(then by induction, if previous two steps are formal polynomial, then the $n+1$th step is also formal polynomial. The proof is just tedious and complex computations. we skip it) But for the to help you understand let me just present a few terms in the beginning.

$$: \varphi(x)^0 := 1, \qquad : \varphi(x)^1 := \varphi(x)$$
$$: \varphi(x)^2 := \varphi(x)^2 - C \qquad : \varphi(x)^3 := \varphi(x)^3 - 3C\varphi(x)$$

@@@@@@@@@@@@@@@@@@@@@@@@@ Skip the proof
For $n = 0$ and $n = 1$, it is trivial that

$$: \varphi(x)^0 := 1, \qquad : \varphi(x)^1 := \varphi(x)$$

We note that from the definition of normal ordering

$$: \varphi(x)^n :=: (\varphi^-(x) + \varphi^+(x))^n := \sum_{k=0}^{n} \binom{n}{k} \varphi^-(x)^k \varphi^+(x)^{n-k}$$

$\varphi^\pm$ satisfies

$$\begin{aligned}
[\varphi^+(x), \varphi^-(x)] &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} e^{i(x,q-p)} [a(\mathbf{p}), a^\dagger(\mathbf{q})] \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} (2\pi)^3 \delta(\mathbf{q} - \mathbf{p}) \\
&= \left(\int_{\mathbb{R}^3} \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \right) 1 = C1
\end{aligned}$$

To prove the claim, we compute

$$* =: \varphi(x)^n : \varphi(x) = \sum_{k=0}^{n} \binom{n}{k} \varphi^-(x)^k \varphi^+(x)^{n-k+1} + \sum_{k=0}^{n} \binom{n}{k} \varphi^-(x)^k \varphi^+(x)^{n-k} \varphi^-(x)$$

We observe the second term.

$$\begin{aligned}
&\varphi^-(x)^k \varphi^+(x)^{n-k} \varphi^-(x) \qquad (\pm \varphi^-(x)^k \varphi^+(x)^{n-k-1} \varphi^-(x) \varphi^+(x)) \\
&\quad = \varphi^-(x)^k \varphi^+(x)^{n-k-1} \varphi^-(x) \varphi^+(x) + \varphi^-(x)^k \varphi^+(x)^{n-k-1} \underbrace{\left(\varphi^+(x) \varphi^-(x) - \varphi^-(x) \varphi^+(x) \right)}_{=C} \\
&\quad = \varphi^-(x)^k \varphi^+(x)^{n-k-2} \varphi^-(x) \varphi^+(x)^2 + 2C \varphi^-(x)^k \varphi^+(x)^{n-k-1} \\
&\quad = \varphi^-(x)^{k+1} \varphi^+(x)^{n-k} + (n-k) C \varphi^-(x)^k \varphi^+(x)^{n-k-1}
\end{aligned}$$

Substituting it in $(*)$, we have

$$\begin{aligned}
* &= \sum_{k=0}^{n} \binom{n}{k} \varphi^-(x)^k \varphi^+(x)^{n-k+1} + \sum_{k=0}^{n} \binom{n}{k} \left(\varphi^-(x)^{k+1}\varphi^+(x)^{n-k} + C(n-k)\varphi^-(x)^k\varphi^+(x)^{n-k-1}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \varphi^-(x)^k \varphi^+(x)^{n-k+1} + \sum_{k=1}^{n+1} \binom{n}{k-1} \varphi^-(x)^k \varphi^+(x)^{n-k+1} + \sum_{k=0}^{n-1} C(n-k)\binom{n}{k} \varphi^-(x)^k \varphi^+(x)^{n-k-1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} \varphi^-(x)^k \varphi^+(x)^{n-k+1} + \sum_{k=0}^{n-1} Cn\binom{n-1}{k} \varphi^-(x)^k \varphi^+(x)^{n-k-1} \\
&=: \varphi(x)^{n+1} : + Cn : \varphi(x)^{n-1} :
\end{aligned}$$

where we used

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}, \qquad (n-k)\binom{n}{k} = n\binom{n-1}{k}$$

which completes the proof of the claim.
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(Proof of the Proposition 16.1) (Recall that we want to prove that $\mathcal{H}(x) =: \varphi(x)^n :$ is Hamiltonian density.)
We should prove

(1) $:\varphi(t,\mathbf{x})^n := e^{itH_0} : \varphi(0,\mathbf{x})^n : e^{-itH_0}$
(2) $[: \varphi(t,\mathbf{x})^n :, : \varphi(t,\mathbf{y})^n :] = 0$

We first observe that

A (1.1) $$[\varphi(0,\mathbf{x}), \varphi(0,\mathbf{y})] = 0$$

Proof. (Proof for (1.1))

$$\begin{aligned}
[\varphi(0,\mathbf{x}), \varphi(0,\mathbf{y})] &= [\varphi^+(0,\mathbf{x}) + \varphi^-(0,\mathbf{x}), \varphi^+(0,\mathbf{y}) + \varphi^-(0,\mathbf{y})] \\
&= \varphi^+(x)\varphi^+(y) + \varphi^+(x)\varphi^-(y) + \varphi^-(x)\varphi^+(y) + \varphi^-(x)\varphi^-(y) \\
&\quad - \varphi^+(y)\varphi^+(x) - \varphi^+(y)\varphi^-(x) - \varphi^-(y)\varphi^+(x) - \varphi^-(y)\varphi^-(x)
\end{aligned}$$

(x and y should be bold) The first and the fourth terms become zero by cancellation by changing of variable $p \leftrightarrow q$. For the second and the third term, we compute

$$\begin{aligned}
\varphi^+(x)\varphi^-(y) - \varphi^-(y)\varphi^+(x) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{d\mathbf{p}}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\frac{1}{\sqrt{2\omega_\mathbf{q}}} e^{i(\mathbf{x}\cdot\mathbf{p}-\mathbf{y}\cdot\mathbf{q})}[a(\mathbf{p}), a^\dagger(\mathbf{q})] \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{d\mathbf{p}}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\frac{1}{\sqrt{2\omega_\mathbf{q}}} e^{i(\mathbf{x}\cdot\mathbf{p}-\mathbf{y}\cdot\mathbf{q})}(2\pi)^3\delta(\mathbf{q}-\mathbf{p}) \\
&= \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_\mathbf{p}} e^{i(\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))}
\end{aligned}$$

Similarly,

$$\begin{aligned}\varphi^-(x)\varphi^+(y) - \varphi^+(y)\varphi^-(x) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{d\mathbf{p}}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\frac{1}{\sqrt{2\omega_\mathbf{q}}} e^{i(\mathbf{x}\cdot\mathbf{p}-\mathbf{y}\cdot\mathbf{q})}[a^\dagger(\mathbf{p}), a(\mathbf{q})] \\ &= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{d\mathbf{p}}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\frac{1}{\sqrt{2\omega_\mathbf{q}}} e^{i(\mathbf{x}\cdot\mathbf{p}-\mathbf{y}\cdot\mathbf{q})}(2\pi)^3\delta(\mathbf{q}-\mathbf{p}) \\ &= -\int_{\mathbb{R}^3}\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_\mathbf{p}} e^{i(\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))}\end{aligned}$$

□

And recall Proposition 13.1 in Lecture 13 that

$$\varphi(t,\mathbf{x}) = e^{itH_0}\varphi(0,\mathbf{x})e^{-itH_0}$$

So that

$$\varphi(t,\mathbf{x})^k = e^{itH_0}\varphi(0,\mathbf{x})^k e^{-itH_0}$$

Now we use the Lemma 16.1 that the normal ordering of φ^n is formal polynomial $: \varphi(t,\mathbf{x})^n := f(\varphi(t,\mathbf{x}))$.

$$: \varphi(t,\mathbf{x})^n := f(\varphi(t,\mathbf{x})) = e^{itH_0} f(\varphi(0,\mathbf{x}))e^{-itH_0} = e^{itH_0} : \varphi(0,\mathbf{x})^n : e^{-itH_0}$$

which proves (1).
For (2), we compute

$$\begin{aligned}[\varphi(t,\mathbf{x})^m, \varphi(t,\mathbf{y})^k] &= [e^{itH_0}\varphi(0,\mathbf{x})^m e^{-itH_0}, e^{itH_0}\varphi(0,\mathbf{y})^n e^{-itH_0}] \\ &= e^{itH_0}[\varphi(0,\mathbf{x})^m, \varphi(0,\mathbf{y})^n]e^{-itH_0} \\ &= 0\end{aligned}$$

Since $: \varphi(t,\mathbf{x})^n := f(\varphi(t,\mathbf{x}))$,

$$[: \varphi(t,\mathbf{x})^n :, : \varphi(t,\mathbf{y})^n :]$$

can be represented by the linear combination of $[\varphi(t,\mathbf{x})^m, \varphi(t,\mathbf{y})^k]$, which proves (2).

1.3. **16.3 Dyson series for the QFT scattering operator.** If the interaction Hamiltonian is defined from the Hamiltonian density, i.e.

$$H_I = \int_{\mathbb{R}^3} d\mathbf{x}\mathcal{H}(0,\mathbf{x})$$

then we can simplify the Dyson series in QFT.
(The time evolution of interaction Hamiltonian satisfies)

$$\begin{aligned}H_I(t) &= e^{itH_0}H_I(0)e^{-itH_0} \\ &= e^{itH_0}\left(\int_{\mathbb{R}^3} d\mathbf{x}\mathcal{H}(0,\mathbf{x})\right)e^{-itH_0} \\ &= \int_{\mathbb{R}^3} d\mathbf{x} e^{itH_0}\mathcal{H}(0,\mathbf{x})e^{-itH_0} \\ &= \int_{\mathbb{R}^3} d\mathbf{x}\mathcal{H}(t,\mathbf{x})\end{aligned}$$

(Only QM case, the each integral of dyson series is written as $\int_{-\infty}^{\infty} dt H_I(t)$. Therefore substituting the above computation, we have)

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-ig)^n}{n!} \int_{\mathbb{R}^{1,3}} \cdots \int_{\mathbb{R}^{1,3}} dx_1 \cdots dx_n T\mathcal{H}(x_1) \cdots \mathcal{H}(x_n)$$

(Q. where the commutation property is used?)

For the simplicity we only consider the second order approximation term:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} TH_I(\theta_1)H_I(\theta_2) d\theta_2 d\theta_1$$

By using the Hamiltonian density, we can write

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left(T \int_{\mathbb{R}^3} d\mathbf{x_1} \mathcal{H}(t_1, \mathbf{x_1}) \int_{\mathbb{R}^3} d\mathbf{x_2} \mathcal{H}(t_2, \mathbf{x_2}) \right) dt_2 dt_1$$

We want to extract the dx integral out of the time ordering operator T. We defined T as follows:

$$TH_I(\theta_1)H_I(\theta_2) \cdots H_I(\theta_n) = H_I(\theta_{\sigma(1)})H_I(\theta_{\sigma(2)}) \cdots H_I(\theta_{\sigma(n)})$$

where $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$. So that when $t_1 = t_2$, the problem occur. By the definition of T, for the region $t_1 \geq t_2$,

$$\int_{t_1 \geq t_2} \int_{\mathbb{R}^3} d\mathbf{x_1} \mathcal{H}(t_1, \mathbf{x_1}) \int_{\mathbb{R}^3} d\mathbf{x_2} \mathcal{H}(t_2, \mathbf{x_2}) dt_2 dt_1$$

For the region $t_1 \leq t_2$,

$$\int_{t_1 \leq t_2} \int_{\mathbb{R}^3} d\mathbf{x_2} \mathcal{H}(t_2, \mathbf{x_2}) \int_{\mathbb{R}^3} d\mathbf{x_1} \mathcal{H}(t_1, \mathbf{x_1}) dt_2 dt_1$$

For a specific region, the integral should be the same. That is,

$$\int_{t_1 = t_2} \int_{\mathbb{R}^3} d\mathbf{x_1} \mathcal{H}(t_1, \mathbf{x_1}) \int_{\mathbb{R}^3} d\mathbf{x_2} \mathcal{H}(t_2, \mathbf{x_2}) dt_2 dt_1 = \int_{t_1 = t_2} \int_{\mathbb{R}^3} d\mathbf{x_2} \mathcal{H}(t_2, \mathbf{x_2}) \int_{\mathbb{R}^3} d\mathbf{x_1} \mathcal{H}(t_1, \mathbf{x_1}) dt_2 dt_1$$

Because of the equal time commutation, the above equation is true.

If the equal time commutation property does not hold, then the following quantities may not equal:

$$\int_{t_1 = t_2} dt_2 dt_1 \int_{\mathbb{R}^3} d\mathbf{x_1} \int_{\mathbb{R}^3} d\mathbf{x_2} \mathcal{H}(t_1, \mathbf{x_1}) \mathcal{H}(t_2, \mathbf{x_2}) \neq \int_{t_1 = t_2} dt_2 dt_1 \int_{\mathbb{R}^3} d\mathbf{x_1} \int_{\mathbb{R}^3} d\mathbf{x_2} \mathcal{H}(t_2, \mathbf{x_2}) \mathcal{H}(t_1, \mathbf{x_1})$$

Thus the time ordering operator become ambiguous.

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