QFT LECTURE NOTE 13-16

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0.1. **Brief review.**

- Lec 1-4
  (1) QM does not fit well with special relativity.
  (2) QM does not allow the creation or annihilation of particles.
  (This is why we study the quantum field theory)
- Lec 5-8
  (1) Bosonic Fock space

$$\mathcal{B} = \{(\psi_0, \psi_1, \cdots) : \psi_n \in \mathcal{H}_{sym}^{\otimes n}, \sum_{n=0}^{\infty} \|\psi_n\|^2 < \infty\}$$

  where $\mathcal{H}_{sym}^{\otimes n}$ is subspace of $L^2(\mathbb{R}^n)$ which element is invariant under the change of the index number. $\mathcal{H}_{sym}^{\otimes n} = \{\psi \in L^2(\mathbb{R}^n), \psi(x_1, \cdots, x_n) = \psi(x_{\sigma(1)}, \cdots, x_{\sigma(n)}) \quad \textit{for all} \quad \sigma \in S_n\}$
  (2) Creation $a^\dagger(x)$, annihilation operator $a(x)$ for non-rela
- Lec 9-12
  Mass shell $X_m = \{p \in \mathbb{R}^{1,3} : p^2 = m^2, p^0 \geq 0\}$
  Measure λ_m: $\int_{X_m} d\lambda_m f(p) = \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{p}} f(\omega_\mathbf{p}, \mathbf{p})$
  Creation $a^\dagger(x)$, annihilation operator $a(x)$ for rela

$$A(f) = \int_{X_m} d\lambda_m f^*(p) a(p), \qquad A^\dagger(f) = \int_{X_m} d\lambda_m f(p) a^\dagger(p)$$

$$a(\mathbf{p}) = \frac{1}{\sqrt{2\omega_\mathbf{p}}} a(p), \qquad a^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2\omega_\mathbf{p}}} a^\dagger(p)$$

  Rela Hamiltonian on Fock space

$$H_0 \psi(p_1, \cdots, p_n) = \left(\sum_{j=1}^{n} p_j^0 \right) \psi(p_1, \cdots, p_n)$$

  Massive scalar free field (and in last lecture, the following massive scalar free field suddenly appeared!)

$$\varphi(x) = \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \left(e^{-i(x,p)} a(\mathbf{p}) + e^{i(x,p)} a^\dagger(\mathbf{p}) \right)$$

  Hyun-Jin Ahn computed that it is related to the Hamiltonian by very complex computation.

@@@@@@@@@@@@@@@@@@ (@: skip)

$$H_0 = \frac{1}{2}\int_{\mathbb{R}^3} d\mathbf{x} : \left((\partial_t \varphi(x))^2 + \sum_{\nu=1}^{3} (\partial_\nu \varphi(x))^2 + m^2 \varphi(x)^2 \right)$$

@@@@@@@@@@@@@@@@@@@@

I also wondered why the author introduced the "massive scalar free field". Maybe it is the most important object since it is already introduced in Lecture 1 (page 2). First of all the scalar field is a field that gives a scalar quantity at each position $x \in \mathbb{R}^3$. One good example is temperature. The solution of the heat equation gives an information of scalar quantity heat on every position and time (t, x). Another good example is pressure. So the massive scalar free field gives a scalar quantity at every (t, x). In this time, the four lectures gives the reason why we use the "massive scalar free field". The "massive scalar free field" is very useful to describe the interaction Hamiltonian effected by repulsion force.

1. Lecture 13 Introduction to φ^4 theory

1.1. **13.1 Evolution of the massive scalar free field.** (We start with how the massive scalar free field evolves.)

prop **Proposition 1.1.** *[Evolution of massive free field] For any $\mathbf{x} \in \mathbb{R}^3$, and $t \in \mathbb{R}$, the massive free field satisfies*

$$\varphi(t, \mathbf{x}) = e^{itH_0} \varphi(0, \mathbf{x}) e^{-itH_0}$$

(To prove that, we need one auxiliary lemma)

Lemma 1.1. *If U is unitary operator on $\mathcal{H}$, extended to $\mathcal{B}$, then for all $f \in \mathcal{H}$,*

$$UA(f)U^{-1} = A(Uf)$$
$$UA^\dagger(f)U^{-1} = A^\dagger(Uf)$$

(Note that e^{itH} is unitary operator. We can expect that this Lemma will be applied as $e^{itH}A(f)e^{-itH} = A(e^{itH}f)$)

Proof. Let $\mathcal{H} = L^2(\mathbb{R})$. We choose $g \in \mathcal{H}^{\otimes n} = L^2(\mathbb{R}^n)$

$$\begin{aligned} L.H.S &= UA(f)U^{-1}g \\ &= U \int_{X_m} d\lambda_m f^*(p) a(p) U^{-1} g(p_1, \cdots, p_n) \\ &= U \int_{X_m} d\lambda_m f^*(p) \sqrt{n} \left[U^{-1}g_1(p_1) \otimes \cdots U^{-1}g_{n-1}(p_{n-1}) \otimes U^{-1}g_n(p) \right] \\ &= \sqrt{n} \left(\int_{X_m} d\lambda_m f^*(p) U^{-1} g_n(p) \right) g_1(p_1) \otimes \cdots \otimes g_{n-1}(p_{n-1}) \end{aligned}$$

Note that

$$\begin{aligned}\int_{X_m} d\lambda_m f^*(p)U^{-1}g_n(p) &= \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}} f^*(\omega_{\mathbf{p}},\mathbf{p})U^{-1}g_n(\omega_{\mathbf{p}},\mathbf{p}) \\ &= \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}} (Uf)^*(\omega_{\mathbf{p}},\mathbf{p})g_n(\omega_{\mathbf{p}},\mathbf{p}) \\ &= \int_{X_m} d\lambda_m (Uf)^*(p)g_n(p)\end{aligned}$$

(Where we applied that the unitary matrix preserves the inner product.)

$$\begin{aligned}L.H.S &= \int_{X_m} d\lambda_m (Uf)^*(p)\sqrt{n}g_1(p_1)\otimes\cdots\otimes g_{n-1}(p_{n-1})\otimes g_n(p) \\ &= A(Uf)g\end{aligned}$$

For $A^\dagger$,

$$\begin{aligned}L.H.S &= UA^\dagger(f)U^{-1}g \\ &= U\int_{X_m} d\lambda_m f(p)a^\dagger(p)U^{-1}g(p_1,\cdots,p_n) \\ &= U\int_{X_m} d\lambda_m f(p)\frac{1}{\sqrt{n+1}}\sum_{j=1}^{n+1}\delta_{p_j}(p)U^{-1}g_1(p_1)\otimes\cdots U^{-1}\hat{g_j}(p_j)\cdots\otimes U^{-1}g_{n+1}(p_{n+1}) \\ &= U\frac{1}{\sqrt{n+1}}\sum_{j=1}^{n+1}U^{-1}g_1(p_1)\otimes\cdots f(p_j)\cdots\otimes U^{-1}g_{n+1}(p_{n+1}) \\ &= \frac{1}{\sqrt{n+1}}\sum_{j=1}^{n+1}g_1(p_1)\otimes\cdots Uf(p_j)\cdots\otimes g_{n+1}(p_{n+1}) \\ &= A^\dagger(Uf) = R.H.S\end{aligned}$$

(We need assumption that $p_i \in X_m$)

(We can extend the proof for the bosonic Fock space.) □

(proof of Prop 13.1):

Proof. @@@@@@@@@@@@@ Remind the notations:

$$A(f) = \int_{X_m} d\lambda_m f^*(p)a(p), \qquad A^\dagger(f) = \int_{X_m} d\lambda_m f(p)a^\dagger(p)$$

$$a(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}a(p), \qquad a^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}a^\dagger(p)$$

@@@@@@@@@@@@@

We can write

$$a(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}a(p) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}A(\delta_p)$$

So that

$$\begin{aligned} e^{itH_0}a(\mathbf{p})e^{-itH_0} &= \frac{1}{\sqrt{2\omega_\mathbf{p}}}e^{itH_0}A(\delta_p)e^{-itH_0} \\ &= \frac{1}{\sqrt{2\omega_\mathbf{p}}}A(e^{itH_0}\delta_p) \\ &= \frac{1}{\sqrt{2\omega_\mathbf{p}}}A(e^{itp^0}\delta_p) \\ &= \frac{1}{\sqrt{2\omega_\mathbf{p}}}\int_{X_m} d\lambda_m(e^{itp^0}\delta_p)^* a(p) \\ &= e^{-itp^0}a(\mathbf{p}) \end{aligned}$$

Similarly @@@ proof skip@@@

$$\begin{aligned} e^{itH_0}a^\dagger(\mathbf{p})e^{-itH_0} &= \frac{1}{\sqrt{2\omega_\mathbf{p}}}e^{itH_0}A^\dagger(\delta_p)e^{-itH_0} \\ &= \frac{1}{\sqrt{2\omega_\mathbf{p}}}A^\dagger(e^{itH_0}\delta_p) \\ &= \frac{1}{\sqrt{2\omega_\mathbf{p}}}A^\dagger(e^{itp^0}\delta_p) \\ &= \frac{1}{\sqrt{2\omega_\mathbf{p}}}\int_{X_m} d\lambda_m(e^{itp^0}\delta_p) a^\dagger(p) \\ &= e^{itp^0}a^\dagger(\mathbf{p}) \end{aligned}$$

By definition of φ,

$$\begin{aligned} \varphi(t,\mathbf{x}) &= \int_{\mathbb{R}^3}\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left(e^{-i(x,p)}a(\mathbf{p}) + e^{i(x,p)}a^\dagger(\mathbf{p})\right) \\ &= \int_{\mathbb{R}^3}\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left(e^{-i(tp^0-\mathbf{x}\cdot\mathbf{p})}a(\mathbf{p}) + e^{i(tp^0-\mathbf{x}\cdot\mathbf{p})}a^\dagger(\mathbf{p})\right) \\ &= e^{itH_0}\int_{\mathbb{R}^3}\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left(e^{i\mathbf{x}\cdot\mathbf{p}}a(\mathbf{p}) + e^{-i\mathbf{x}\cdot\mathbf{p}}a^\dagger(\mathbf{p})\right)e^{-itH_0} \\ &= e^{itH_0}\varphi(0,\mathbf{x})e^{-itH_0} \end{aligned}$$

□

1.2. **13.2 φ^4 theory.** (Now we consider the system having interaction!!)

When we consider a system of boson having interaction (repelling each other), In section 8.3, we modified the Hamiltonian

$$H\psi = -\frac{1}{2m}\frac{d^2}{dx^2}\psi + \sum_i V(x_i)\psi + \frac{1}{2}\sum_{i\neq j}W(x_i-x_j)\psi$$

(Moreover the Hamiltonian is formally represented as the following form:)

$$H = \int dx a^\dagger(x) \left(-\frac{1}{2m} \frac{d^2}{dx^2} + V(x) \right) a(x) + \int \int dx dy W(x-y) a^\dagger(x) a^\dagger(y) a(x) a(y)$$

When the interaction $W(x-y)$ approaches $\delta(x-y)$, the last term closes to

$$\int dx a^\dagger(x) a^\dagger(x) a(x) a(x)$$

which is normal ordering of four a's. (creation or annihilation operator).

From now on, we consider the Hamiltonian $H = H_0 + gH_I$ where g is coupling parameter

$$H_I = \frac{1}{4!} \int d\mathbf{x} : \varphi(0, \mathbf{x})^4 :$$

@@@@@@@@@@@@@@@@@@@@@@@ Applying the calculation of Ahn's lecture, we can represent

$$H = \int d\mathbf{x} : \left(\frac{1}{2} (\partial_t \varphi(0, \mathbf{x}))^2 + \frac{1}{2} \sum_{\nu=1}^{3} (\partial_\nu \varphi(0, \mathbf{x}))^2 + \frac{m^2}{2} \varphi(0, \mathbf{x})^2 + \frac{g}{4!} \varphi(0, \mathbf{x})^4 \right) :$$

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We want to understand how states evolve according to the Hamiltonian having interaction. For that we need scattering theory.
Next three lectures consider about scattering.
Lec 14,15 - QM scattering (Non relativistic, Not allow creation, annihilation)
Lec 16 - QFT scattering

2. Leture 14 Scattering

2.1. **14.1 Classical scattering.** Let me explain about scattering. Scattering is just that when we put a particle into the potential, then what is the outgoing state? Without potential, if we put a particle with (x, v) then the particle evolves following a trajectory $(x + vt, v)$. Some good examples of non-quantum scattering is like this:(Draw picture) (1) Elastic collision with hard sphere. (2) Attraction (3) Repulsion. We are interested in the repulsion case. We want to define a scattering operator S. When we input a particle at (x, v) how can we define S which gives outgoing particle at state (y, u)?

We assume that the particle is at $(\mathbf{x}', \mathbf{v}')$ when $t = 0$.
For $t < 0$ we define $(\mathbf{x}, \mathbf{v})_t := (\mathbf{x}', \mathbf{v}')$.
If the limit exists, then we define

$$\Omega_+(\mathbf{x}, \mathbf{v}) = \lim_{t \to -\infty} (\mathbf{x}, \mathbf{v})_t$$

This operator means the (position,velocity) of particle far past.

For $t > 0$, we define $(\mathbf{x}, \mathbf{v})_t := (\mathbf{x}', \mathbf{v}')$. If the limit exists, we define

$$\Omega_-(\mathbf{x}, \mathbf{v}) = \lim_{t \to \infty} (\mathbf{x}, \mathbf{v})_t$$

This operator means the (position,velocity) of particle far future.
For given $(\mathbf{x}, \mathbf{v})$, Ω_+ and Ω_- gives informations about the particle position and velocity at $t = 0$.

We define the scattering operator as:

$$S := \Omega_-^{-1}\Omega_+$$

Let ($\mathbf{y}$,$\mathbf{u}$) = S($\mathbf{x}$,$\mathbf{v}$) then

$$\Omega_-(\mathbf{y},\mathbf{u}) = \Omega_+(\mathbf{x},\mathbf{v})$$

Scattering operator S means that when we put a particle with $(\mathbf{x},\mathbf{v})$ on the potential we obtain scattering state $S(\mathbf{x},\mathbf{v}) = (\mathbf{y},\mathbf{u})$.

2.2. **14.2 Scattering in non-relativistic QM.** Now we consider the Hamiltonian having an interaction potential:

$$H = H_0 + gV$$

We define

$$U(t) = e^{-itH}, \qquad U_0(t) = e^{-itH_0}$$

(The lower indice H_0 does not mean the relativistic case.)
Let $|\psi\rangle$ is state of system at $t = 0$. (Draw picture)
Free evolution of classical particle with initial data $(\mathbf{x},\mathbf{v})$: $(\mathbf{x}+\mathbf{v}t)_{t\in\mathbb{R}}$
Free evolution of quantum particle with initial state $|\psi\rangle$: $(U_0(t)|\psi\rangle)_{t\in\mathbb{R}}$
This is called "the trajectory of $|\psi\rangle$.
For $t < 0$,
Far past: $U_0(t)|\psi\rangle$.
Affected by H: $U(-t)U_0(t)|\psi\rangle$
(Draw picture)
We define

$$\Omega_+|\psi\rangle = \lim_{t\to-\infty} U(-t)U_0(t)|\psi\rangle$$

For $t > 0$ we define

$$\Omega_-|\psi\rangle = \lim_{t\to\infty} U(-t)U_0(t)|\psi\rangle$$

Scattering operator as:

$$S := \Omega_-^{-1}\Omega_+$$

$S|\psi\rangle$ means "trajectory of a particle in the far future if it is on the trajectory $|\psi\rangle$ in the far past.
If $\Omega_-|\varphi\rangle = \Omega_+|\psi\rangle$, (Draw picture)
(It is hard to consider Ω_-^{-1}. We re-define the scattering operator)

$$S = \Omega_-^{-1}\Omega_+ = \lim_{t\to\infty}\lim_{t_0\to-\infty} U_0(-t)U(t-t_0)U_0(t_0)$$

Two main problems:

- Limits may not exist in the definition of Ω_+ and Ω_-.
- We need $Range(\Omega_+) \subset Range(\Omega_-)$ to define S.

(Q. It seems necessary surjectivity of Ω_-) We just define Ω_-^{-1} by the following way:

$$\Omega_-^{-1}\psi\rangle = \lim_{t\to\infty} U_0(-t)U(t)|\psi\rangle$$

If the second condition is not satisfied, then the particle can be "trapped by the potential".

Theorem 2.1. *Let $\mathcal{H} = L^2(\mathbb{R}^3)$, with the one-particle free Hamiltonian $H_0 = -\frac{1}{2m}\triangle$ and $H_I = V \in L^2(\mathbb{R}^3)$. Then, $H = H_0 + V$ is essesntially self-adjoint, and Ω_+ and Ω_- exist as operators in $L^2(\mathbb{R}^3)$.*

(If H_I is in L^2, then we can safely use the scattering operator)

2.3. **14.3 Dyson series expansion.** (Dyson series expansion is formal power series expansion of the scattering operator.)
Let $H = H_0 + gH_I$. Fix $t_0 < 0$, we define

$$G(t) = U_0(-t)U(t - t_0)U_0(t_0)$$

where $U(t) = e^{-itH}$ and $U_0(t) = e^{-itH_0}$. Then

$$\begin{aligned}\frac{d}{dt}G(t) &= \frac{d}{dt}\{e^{itH_0}e^{-i(t-t_0)H}e^{-itH_0}\} \\ &= iH_0(e^{itH_0}e^{-i(t-t_0)H}e^{-itH_0}) + e^{itH_0}(-iH)e^{-i(t-t_0)H}e^{-itH_0} \\ &= i(H_0 - H)G(t) \\ &= -igH_I(t)G(t).\end{aligned}$$

(From Section 2.6, we observed that the Hamiltonian commutes with unitary operator:)

$$\frac{d}{dt}U(t) = -iHU(t) = -iHe^{-itH} = -iU(t)H = ie^{-itH}H.$$

$$\begin{aligned}G(t) &= G(t_0) + \int_{t_0}^{t} d\theta_1 G'(\theta_1) \\ &= 1 - ig\int_{t_0}^{t} d\theta_1 H_I(\theta_1)G(\theta_1)\end{aligned}$$

Once we rewrite $G(\theta_1)$ on the time integral as

$$G(\theta_1) = 1 - ig\int_{t_0}^{\theta_1} d\theta_2 H_I(\theta_2)G(\theta_2)$$

We iteratively have

$$\begin{aligned}G(t) &= 1 - ig\int_{t_0}^{t} d\theta_1 H_I(\theta_1)\left(1 - ig\int_{t_0}^{\theta_1} d\theta_2 H_I(\theta_2)G(\theta_2)\right) \\ &= 1 - ig\int_{t_0}^{t} d\theta_1 H_I(\theta_1) + (-ig)^2\int_{t_0}^{t} d\theta_1\int_{t_0}^{\theta_1} d\theta_2 H_I(\theta_1)H_I(\theta_2)G(\theta_2)\end{aligned}$$

By iteration, one can have

$$\begin{aligned}G(t) &= 1 + \sum_{n=1}^{\infty}(-ig)^n\int_{t_0}^{t}\int_{t_0}^{\theta_1}\cdots\int_{t_0}^{\theta_{n-1}} H_I(\theta_1)H_I(\theta_2)\cdots H_I(\theta_n)d\theta_n\cdots d\theta_1 \\ &= 1 + \sum_{n=1}^{\infty}\frac{(-ig)^n}{n!}\int_{t_0}^{t}\int_{t_0}^{t}\cdots\int_{t_0}^{t} TH_I(\theta_1)H_I(\theta_2)\cdots H_I(\theta_n)d\theta_n\cdots d\theta_1\end{aligned}$$

where (T is ordering operator)

$$TH_I(\theta_1)H_I(\theta_2)\cdots H_I(\theta_n) = H_I(\theta_{\sigma(1)})H_I(\theta_{\sigma(2)})\cdots H_I(\theta_{\sigma(n)})$$

(We can simplify the integral further:)
We note that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$.
@@@@@@@@@@@@@@@ For simple example see:

$$\int_0^1 1dx = 1, \quad \int_0^1 \int_0^x 1dydx = \frac{1}{2!}, \quad \int_0^1 \int_0^x \int_0^y 1dzdydx = \frac{1}{3!}$$

or..

$$\int_{-\infty}^{\infty} \int_{-\infty}^{x} f_1(x) f_2(y) dydx = \int_{-\infty}^{\infty} \int_{x}^{\infty} f_1(-x) f_2(-t) dtdx$$

@@@@@@@@@@@@@@@ Taking limit, we have

$$\begin{aligned} S &= \lim_{t\to\infty} \lim_{t_0 \to -\infty} G(t) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-ig)^n}{n!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} TH_I(\theta_1) H_I(\theta_2) \cdots H_I(\theta_n) d\theta_n \cdots d\theta_1 \end{aligned}$$

Wiki: This series diverges asymptotically, but in quantum electrodynamics (QED) at the second order the difference from experimental data is in the order of 10^{-10}.

3. Lecture 15 Born approximation

3.1. **15.1 Derivation of the Born approximation.** The Born approximation is first order approximation of the scattering operator S. We just want to compute the scattering operator by acting on a real momentum state such as $|\mathbf{p_1}\rangle$.

We consider a first order approximation of scattering operator S:

$$S = 1 + (-ig) \int_{-\infty}^{\infty} dt H_I(t) + O(g^2)$$

where $H_I(t) = e^{itH_0} H_I e^{-itH_0}$. (I don't know why we can write this.)
Let $H_0 = -\frac{1}{2m}\triangle$ and $H_I = V \in L^2(\mathbb{R}^3)$. (Then there exists the limit of $\Omega_\pm$).
Let us compute $\langle \mathbf{p_2}|S|\mathbf{p_1}\rangle$. (The meaning is "probability of momentum of the outgoing particle is p_2 when we input particle has momentum p_1.")

$$\langle \mathbf{p_2}|S|\mathbf{p_1}\rangle = \underbrace{\langle \mathbf{p_2}|1|\mathbf{p_1}\rangle}_{:=I} + (-ig) \int_{-\infty}^{\infty} dt \underbrace{\langle \mathbf{p_2}|e^{itH_0} H_I e^{-itH_0}|\mathbf{p_1}\rangle}_{:=II} + O(g^2)$$

See the definition of $|\mathbf{p_1}\rangle$ in Section 4.9

$$|\mathbf{p_1}\rangle(x) = e^{i\mathbf{p_1}\cdot\mathbf{x}}, \qquad |\mathbf{p_1}\rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p_1})$$

So that

$$I = \langle \mathbf{p_2}|1|\mathbf{p_1}\rangle = \langle \mathbf{p_2}|\mathbf{p_1}\rangle = (2\pi)^3 \delta^3(\mathbf{p_2} - \mathbf{p_1})$$

To compute II, (we first consider)

$$\begin{aligned} e^{-itH_0}|\mathbf{p_1}\rangle &= e^{-it|\mathbf{p}|^2/2m} |(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p_1}) \\ &= e^{-it|\mathbf{p_1}|^2/2m} |\mathbf{p_1}\rangle \end{aligned}$$

Then by an explicit computation,

$$II = e^{it(|\mathbf{p_2}|^2 - |\mathbf{p_1}|^2)/2m} \langle \mathbf{p_2}|V|\mathbf{p_1}\rangle$$

Since the potential $V(x)$ is in spatial space, $V \in L^2(\mathbb{R}^3_x)$, we should compute it in spatial space. By definition of $|\mathbf{p_1}\rangle$ in spatial space, we have

$$\langle \mathbf{p_2}|V|\mathbf{p_1}\rangle = \int_{\mathbb{R}^3} d\mathbf{x^3} e^{-i\mathbf{x}\cdot(\mathbf{p_2}-\mathbf{p_1})} V(x)dx = \hat{V}(\mathbf{p_2}-\mathbf{p_1})$$

Combining I and II, we get

$$\begin{aligned}\langle \mathbf{p_2}|S|\mathbf{p_1}\rangle &= (2\pi)^3\delta^3(\mathbf{p_2}-\mathbf{p_1}) + (-ig)\hat{V}(\mathbf{p_2}-\mathbf{p_1})\int_{-\infty}^{\infty} dt e^{it(|\mathbf{p_2}|^\mathbf{2}-|\mathbf{p_1}|^\mathbf{2})/2m} + O(g^2) \\ &= (2\pi)^3\delta^3(\mathbf{p_2}-\mathbf{p_1}) + (-ig)(2\pi)\hat{V}(\mathbf{p_2}-\mathbf{p_1})\delta((|\mathbf{p_2}|^\mathbf{2}-|\mathbf{p_1}|^\mathbf{2})/2m) + O(g^2)\end{aligned}$$

This is called Born approximation.

3.2. **15.2 What does it mean?** But the momentum state $|\mathbf{p_1}\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p_1})$ is not proper. To find the meaning of Born approximation, we instead use a proper state $|\psi_\epsilon\rangle$:

$$|\psi_\epsilon\rangle = \frac{1}{(2\pi\epsilon)^{3/2}} e^{-(\mathbf{p}-\mathbf{p_1})^2/2\epsilon}$$

@@@@@@@@@@@@@@ We re-compute $\langle \mathbf{p_2}|S|\psi_\epsilon\rangle$ in the same way.

$$I = \langle \mathbf{p_2}|1|\psi_\epsilon\rangle = \langle \mathbf{p_2}|\psi_\epsilon\rangle = \frac{C_1}{\epsilon^{3/2}} e^{-(\mathbf{p_2}-\mathbf{p_1})^2/2\epsilon}$$

To compute II, we take fourier transform to deal with $|\psi_\epsilon\rangle$ as a spatial variable state.

$$|\psi_\epsilon\rangle(x) = \int_{\mathbb{R}^3} e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{(2\pi\epsilon)^{3/2}} e^{-(\mathbf{p}-\mathbf{p_1})^2/2\epsilon}$$

$$II = e^{it(|\mathbf{p_2}|^\mathbf{2}-|\mathbf{p_1}|^\mathbf{2})/2m}\langle \mathbf{p_2}|V|\mathbf{p_1}\rangle$$

?? I can't compute it..

$$II = C_2(2\pi)\hat{V}(\mathbf{p_2}-\mathbf{p_1})\frac{1}{\sqrt{\epsilon}} e^{-\frac{(|\mathbf{p_2}|^\mathbf{2}-|\mathbf{p_1}|^\mathbf{2})^\mathbf{2}}{8m^2\epsilon}}$$

So that @@@@@@@@@@@@@

(Then by similar computation, we can have)

$$\langle \mathbf{p_2}|S|\psi_\epsilon\rangle = \underbrace{\frac{C_1}{\epsilon^{3/2}} e^{-(\mathbf{p_2}-\mathbf{p_1})^2/2\epsilon}}_{A(\mathbf{p_2})} + \underbrace{(-ig)C_2(2\pi)\hat{V}(\mathbf{p_2}-\mathbf{p_1})\frac{1}{\sqrt{\epsilon}} e^{-\frac{(|\mathbf{p_2}|^\mathbf{2}-|\mathbf{p_1}|^\mathbf{2})^\mathbf{2}}{8m^2\epsilon}}}_{B(\mathbf{p_2})}$$

(The first term is not related to the potential V. The second term implies probability effected by potential.)

Once we define

$$f(\mathbf{p_2}) = (\langle \mathbf{p_2}|S|\psi_\epsilon\rangle)^2$$

Then $f(\mathbf{p_2})$ is proportional to the probability density of the momentum of the outgoing state. (Draw a picture A_ϵ and B_ϵ, and explain it.)

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