Chap 6. Doob-Meyer type decomposition for rough paths

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Introduction
Definition (Continuous semi-martingale)

[RY91] A continuous process \((S_t : t \geq 0)\) on the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) is a \textit{continuous semi-martingale} if it can be written in

\[ S_t = M_t + A_t, \]

where \(M\) is a continuous local-martingale and \(A\) is a continuous adapted process of finite variation.

Notation.

\[ \mathcal{M} := \text{(the space of continuous local martingales)} \]
\[ \mathcal{V} := \text{(the space of continuous adapted process of finite variation)} \]
Assume that \( M, \tilde{M} \in \mathcal{M} \), vanishing at zero, and \( A, \tilde{A} \in \mathcal{V} \) satisfies

\[ M + A \equiv \tilde{M} + \tilde{A}. \]

Then,

\[ M \equiv \tilde{M} \quad \text{and} \quad A \equiv \tilde{A}. \]

Furthermore, if \( S = M + A \equiv 0 \) on some random interval \([0, \tau)\) where \( \tau \) is a stopping time, then \([M] \equiv 0 \) on \([0, \tau)\) and \( A \equiv 0 \) on \([0, \tau)\).
Corollary (Corollary 6.2. in Textbook)

Let $B$ be a $d$-dimensional Brownian motion and let $Y, Z, \tilde{Y}, \tilde{Z}$ be continuous stochastic process adapted to the filtration generated by $B$. Assume that

$$\int_0^T YdB + \int_0^T Zdt \equiv \int_0^T \tilde{Y}dB + \int_0^T \tilde{Z}dt \quad \text{on } [0, T].$$

Then, $Y \equiv \tilde{Y}$ and $Z \equiv \tilde{Z}$ on $[0, T]$. 
**Proof.** It suffices to consider the case $\tilde{Y} = 0$ and $\tilde{Z} = 0$ and show $Y \equiv 0$. By the previous proposition,

$$\left[ \int_0 \cdot Y dB \right] \equiv 0 \Rightarrow \left[ \sum_{k=1}^d \int_0 \cdot Y_k dB^k \right] \equiv 0 \text{ on } [0, T].$$

Note the fact that $d[B^k, B^l]_t = \delta^{kl} dt$. Then, it follows from

$$\left[ \sum_{k=1}^d \int_0 \cdot Y_k dB^k \right] \equiv \sum_{k, l=1}^d \int_0 \cdot Y_k Y_l d[B^k, B^l] = \sum_{k=1}^d \int_0 Y_k^2 dt$$

that $Y \equiv 0$. \qed
**Question.** Suppose

\[
\int_0^t YdX + \int_0^t Zdt \equiv \int_0^t \tilde{Y}dX + \int_0^t \tilde{Z}dt
\]

holds on \([0, \, T]\). Then, under what condition do \(Y \equiv \tilde{Y}\) and \(Z \equiv \tilde{Z}\) holds?

**Answer.** For all \(0 \leq s \leq t \leq T\), one has

\[
\int_s^t (Y - \tilde{Y})dX = \int_s^t (Z_r - \tilde{Z}_r)dr = O(|t - s|)
\]

from the continuity of \(Z\).
For $\alpha \in \left( \frac{1}{3}, \frac{1}{2} \right]$, one can obtain

$$\int_{s}^{t} (Y - \tilde{Y})dX = O(|t - s|^{2\alpha}).$$

Recall the Definition 4.6. in Textbook:

$$Y_{s,t} = Y'_{s}X_{s,t} + O(|t - s|^{2\alpha})$$

and note that

$$I' = Y - \tilde{Y} \quad \text{where} \quad I = \int_{0}^{t} (Y - \tilde{Y})dX$$

Hence, uniqueness of $I'$ implies $Y \equiv \tilde{Y}$. 
Our question turns to under what condition is the uniqueness of Gubinelli derivative guaranteed?
Roughness and its example
Roughness and its example

**Definition (Rough, Definition 6.3. in Textbook)**

For fixed $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ and $s \in [0, T)$, $X \in C^\alpha([0, T], V)$ is **rough at time $s$** if the following relation holds for any $v^* \in V^* \setminus \{0\}$:

$$\limsup_{t \to s^+} \frac{|v^*(X_s, t)|}{|t - s|^{2\alpha}} = \infty.$$ 

And $X$ is **truly rough** if $X$ is rough on some dense subset of $[0, T]$. 
Roughness and its example

**Theorem (Theorem 6.6. in Textbook)**

*With probability one, Brownian motion on $V = \mathbb{R}^d$ is truly rough, relative to any Hölder exponent $\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right)$.*

**Proof.** It suffices to show that for fixed time $s$ and $\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right)$,

$$P \left[ \limsup_{t \to s^+} \frac{\left| \varphi(B_{s,t}) \right|}{|t - s|^{2\alpha}} = \infty, \forall \varphi \in V^*, \left| \varphi \right| = 1 \right] = 1.$$

We will use the following well-known fact that for all $t \geq 0$,

$$P \left[ \limsup_{h \to 0^+} \frac{|B_{t,t+h}|}{\psi(h)} = \sqrt{2} \right] = 1,$$

where $\psi(h) := h^{\frac{1}{2}} \left( \log \log \frac{1}{h} \right)^{\frac{1}{2}}$. 
Roughness and its example

Then, there exists $c > 0$ such that for every fixed unit dual vector $\varphi \in V^*$ and every fixed $s \in [0, T)$,

$$P \left[ \limsup_{t \to s^+} \frac{|\varphi(B_{s,t})|}{\psi(t - s)} \geq c \right] = 1.$$

Take $K \subset V^*$ to be dense, countable subset of unit dual vectors. Since $K$ is countable, one has

$$P \left[ \limsup_{t \to s^+} \frac{|\varphi(B_{s,t})|}{\psi(t - s)} \geq c, \, \varphi \in K \right] = 1,$$

and due to the density of $K$, every unit dual vector $\varphi \in V^*$ is the limit of some $(\varphi_n) \subset K$. 
Then,

\[ c \leq \limsup_{t \to s^+} \frac{|\varphi_n(B_s,t)|}{\psi(t-s)} \leq \limsup_{t \to s^+} \frac{|\varphi(B_s,t)|}{\psi(t-s)} + |\varphi_n - \varphi| \nu^* \limsup_{t \to s^+} \frac{|B_s,t|}{\psi(t-s)} \]

almost surely and we let \( n \) go to infinity to obtain

\[ c \leq \limsup_{t \to s^+} \frac{|\varphi(B_s,t)|}{\psi(t-s)} \text{ almost surely.} \]

Hence, we can take \((t_n)\) converging to \( s \) such that

\[
\frac{|\varphi(B_s,t_n)|}{|t_n - s|^{2\alpha}} = \frac{|\varphi(B_s,t_n)|}{\psi(t_n-s)} \frac{\psi(t_n-s)}{|t_n - s|^{2\alpha}} \\
\geq (c - \frac{1}{n})|t_n - s|^{\frac{1}{2} - 2\alpha} \left( \log \log \frac{1}{t_n - s} \right)^{\frac{1}{2}} \to \infty
\]

for almost every sample path. \( \Box \)
Doob-Meyer for rough paths
Proposition (Proposition 6.4. in Textbook)

(Uniqueness of $Y'$) Let $X = (X, \mathbb{X}) \in \mathcal{C}^\alpha, (Y, Y') \in \mathcal{D}_X^{2\alpha}$, so that the rough integral $\int Y dX$ exists. Suppose $X$ is rough at some time $s$. Then,

$$Y_{s,t} = O(|t-s|^{2\alpha}) \quad \text{as} \quad t \rightarrow s+ \quad \implies \quad Y'_s = 0.$$

As a consequence, if $X$ is truly rough and $(Y, \tilde{Y}') \in \mathcal{D}_X^{2\alpha}$ is another controlled rough path, then $Y' \equiv \tilde{Y}'$.

Proof. From the definition of $(Y, Y')$, one have

$$\frac{Y'_s X_{s,t}}{|t-s|^{2\alpha}} = \frac{Y_{s,t}}{|t-s|^{2\alpha}} + O(1) = O(1), \quad s < t < s + \epsilon.$$
For every \( w^* \in \overline{W}^* \), the map

\[
V \ni v \mapsto w^*(Y'_s v)
\]

defines an element \( v^* \in V^* \) so that

\[
\frac{|v^*(X_s,t)|}{|t-s|^{2\alpha}} = \left| w^* \left( \frac{Y'_s X_s,t}{|t-s|^{2\alpha}} \right) \right| = O(1) \quad \text{as } t \to s + .
\]

Since \( X \) is rough at \( s \), we have \( v^* = 0 \), which implies that \( Y'_s = 0 \). \( \square \)
Theorem (Theorem 6.5. in Textbook)

(Doob-Meyer for rough paths) Suppose that $X$ is rough at some time $s \in [0, T)$ and let $(Y, Y') \in D^2_X$. Then,

$$\int_s^t YdX = O(|t - s|^{2\alpha}) \quad \text{as } t \to s+ \implies Y_s = 0.$$

As a consequence, if $X$ is truly rough, $(\tilde{Y}, \tilde{Y}') \in D^2_X$ and $Z, \tilde{Z} \in C([0, T], W)$, then the identity

$$\int_0^t YdX + \int_0^t Zdt \equiv \int_0^t \tilde{Y}dX + \int_0^t \tilde{Z}dt$$

on $[0, T]$ implies that $(Y, Y') \equiv (\tilde{Y}, \tilde{Y}')$ and $Z \equiv \tilde{Z}$ on $[0, T]$. 
Quantitative version of roughness
A path $X : [0, T] \rightarrow V$ with values in a Banach space $V$ is said to be \( \theta \)-Hölder rough for $\theta \in (0, 1)$, on scale (smaller than) $\epsilon_0 > 0$, if there exists a constant $L := L_\theta(X) := L(\theta, \epsilon_0, T; X) > 0$ such that for every $\varphi \in V^*$, $s \in [0, T]$ and $\epsilon \in (0, \epsilon_0]$ there exists $t \in [0, T]$ such that

$$ |t - s| < \epsilon \quad \text{and} \quad |\varphi(X_{s,t})| \geq L_\theta(X)\epsilon^\theta |\varphi|. $$

The largest such value of $L$ is called the \textit{modulus of \( \theta \)-Hölder roughness} of $X$.

\textbf{Remark.} Every element in $C^\alpha$ which is \( \theta \)-Hölder rough for $\theta < 2\alpha$ is truly rough.
Proposition (Proposition 6.8. in Textbook)

Let \((X, X) \in C^\alpha([0, T], V)\) be such that \(X\) is \(\theta\)-Hölder rough for some \(\theta \in (0, 1]\). Then, for every controlled rough path \((Y, Y') \in D^2_\alpha([0, T], W)\) one has,

\[
L \epsilon^\theta \|Y'\|_\infty \leq \text{osc}(Y, \epsilon) + \|R^Y\|_{2\alpha} \epsilon^{2\alpha}.
\]

As a consequence, if \(\theta < 2\alpha\), \(Y'\) is uniquely determined from \(Y\).

Remark. This Proposition is quantitative version of Proposition 6.4. in Textbook.
Quantitative version of roughness

**Proof.** Fix \( s \in [0, T] \) and \( \epsilon \in (0, \epsilon_0] \). From the definition of Gubinelli derivative, one has

\[
\sup_{|t-s| \leq \epsilon} |Y'_s X_{s,t}| \leq \sup_{|t-s| \leq \epsilon} (|Y_{s,t}| + |R^Y_{s,t}|) \leq \text{osc}(Y, \epsilon) + \|R^Y\|_{2\alpha} \epsilon^{2\alpha}.
\]

Since \( X \) is \( \theta \)-Hölder rough, for unit \( \varphi \in W^* \), there exists \( t_0 = t_0(\varphi) \) with \( |t_0 - s| < \epsilon \) such that

\[
|\varphi(Y'_s X_{s,t_0})| = |((Y'_s)^* \varphi)(X_{s,t_0})| > L\epsilon^\theta |(Y'_s)^* \varphi|.
\]

Hence, from the arbitrarity of \( \varphi \) and \( s \), one can obtain

\[
L\epsilon^\theta |Y'| \leq \text{osc}(Y, \epsilon) + \|R^Y\|_{2\alpha} \epsilon^{2\alpha}.
\]
Let \((X, \mathbb{X}) \in C^\alpha([0, T], V)\) be such that \(X\) is \(\theta\)-Hölder rough with \(\theta < 2\alpha\). Let \((Y, Y') \in D^2_X([0, T], \mathbb{W})\) and \(Z \in C^\alpha([0, T], W)\) and set

\[
I_t := \int_0^t Y_s dX_s + \int_0^t Z_s ds.
\]

Then, there exist constants \(r > 0\) and \(q > 0\) such that

\[
\|Y\|_\infty + \|Z\|_\infty \leq MR^q \|I\|_\infty^r,
\]

where constant \(M\) only depends on \(\alpha, \theta\) and \(T\) and

\[
R := 1 + L\theta(X)^{-1} + \|X\|_\alpha + \|Y, Y'\|_{X;2\alpha} + |Y_0| + |Y'_0| + \|Z\|_\alpha + |Z_0|.
\]
Summary
1. $X$ is rough at time $s$ \iff \limsup_{t \to s^+} \frac{|\nu^*(X_{s,t})|}{|t - s|^{2\alpha}} = \infty$, $\forall \nu^* \in V^*$

2. Brownian motion is truly rough.

3. Roughness of $X$ implies the uniqueness of Gubinelli derivative and therefore, the uniqueness of Doob-Meyer type decomposition.
THANK YOU FOR LISTENING