Chap 4. Integration against rough paths

Choi, Min-Jun

Seoul National University

minjunchoi@snu.ac.kr

April 2, 2020
Overview

1 Introduction
   - Young Integral

2 Integration of 1-form
   - Sewing Lemma

3 Integration of controlled rough paths
   - The general rough path integral
The aim of this chapter is to give a meaning to the expression $\int Y_t dX_t$ a suitable class of integrands $Y$, integrated against a rough path $X$. 
Let $V$ be a Banach space and $\alpha + \beta > 1$. If $X \in C^\alpha ([0, T], V)$ and $Y \in C^\beta ([0, T], V)$, then the Young integral exists, that is,

$$
\int_0^1 Y_t dX_t = \lim_{|\mathcal{P}| \to 0} \sum_{[s, t] \in \mathcal{P}} Y_s X_{s, t},
$$

where $\mathcal{P}$ denotes a partition of $[0, 1]$ and $|\mathcal{P}|$ denotes the length of the largest element of $\mathcal{P}$. Moreover Young integral has the bound

$$
\left| \int_s^t Y_r dX_r - Y_s X_{s, t} \right| \leq C \|Y\|_\beta \|X\|_\alpha |t - s|^\alpha + \beta.
$$ (1)

Here, our main interest is to reduce $\alpha > \frac{1}{2}$ in the case $\alpha = \beta$. The easiest way for a function $Y$ to "looks like $X$" is to have $Y_t = F(X_t)$ for some sufficiently smooth $F : V \to \mathcal{L}(V, W)$, called a 1-form.
Let $Y = F(X)$ and $X = (X, \overline{X}) \in \mathcal{C}^\alpha([0, T], V)$. When $F : V \to \mathcal{L}(V, W)$ is in $\mathcal{C}_b^2$, a Taylor approximation gives

$$F(X_r) \approx F(X_s) + DF(X_s)X_{s,r}$$

for $r$ in some (small) interval $[s, t]$. Then since the definition of Young integral, $F(X_r) \approx F(X_s)$ for $r \in [s, t]$ and then

$$\int_0^1 F(X_s) dX_s \approx \sum_{[s,t] \in P} [F(X_s) X_{s,t} + DF(X_s) \overline{X}_{s,t}],$$

which is a good enough approximation.
Why should this be good enough?
The intuition is as follows: given $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, neither $|X_{s,t}| \sim |t - s|^\alpha$ nor $|X_{s,t}| \sim |t - s|^{2\alpha}$ in the above sum will be negligible as $|\mathcal{P}| \to 0$.
Moreover, $X^{(3)}_{s,t} \sim |t - s|^{3\alpha} = o(|t - s|)$. So, we will see that this limit,

$$\int_0^1 F(X_s) dX_s = \lim_{|\mathcal{P}| \to 0} \sum_{[s,t] \in \mathcal{P}} \left[ F(X_s) X_{s,t} + DF(X_s) X_{s,t} \right], \quad (2)$$

does exists and call it rough integral.
Lemma 4.1

Let $F : V \to \mathcal{L}(V, W)$ be a $C^2_b$ function and let $(X, \dot{X}) \in \mathcal{C}^\alpha$ for some $\alpha > \frac{1}{3}$. Set $Y_s := F(X_s)$, $Y'_s := DF(X_s)$ and $R^Y_{s,t} := Y_{s,t} - Y'_sX_{s,t}$. Then

$$Y, Y' \in C^\alpha \text{ and } R^Y \in C^{2\alpha}.$$ 

More precisely, we have the estimates

$$\|Y\|_\alpha \leq \|DF\|_\infty \|X\|_\alpha,$$

$$\|Y'\|_\alpha \leq \|D^2F\|_\infty \|X\|_\alpha,$$

$$\|R^Y\|_{2\alpha} \leq \frac{1}{2} \|D^2F\|_\infty \|X\|_{2\alpha}.$$
Proof. $C^2_b$ regularity of $F$ implies that $F$ and $DF$ are both Lipschitz continuous with Lipschitz constants $\|DF\|_\infty$ and $\|D^2F\|_\infty$ respectively. The $\alpha$-Hölder bounds on $Y$ and $Y'$ are then immediate. For the remainder term, consider the function

$$[0, 1] \ni \xi \mapsto F(X_s + \xi X_{s,t}).$$

A Taylor expansion, with intermediate value remainder, yields $\xi \in (0, 1)$ such that

$$R^Y_{s,t} = F(X_t) - F(X_s) - DF(X_s)X_{s,t} = \frac{1}{2} D^2F(X_s + \xi X_{s,t})(X_{s,t}, X_{s,t}).$$

The claimed $2\alpha$-Hölder estimae, in the sense that $|R^Y_{s,t}| \lesssim |t - s|^{2\alpha}$, then follows at once. \[\square\]
Let $Z_t := \int_0^t Y_r dX_r$. Then by Young’s inequality (1), one has

$$Z_{s,t} = Y_s X_{s,t} + o(|t - s|).$$

That is, the function $Ξ_{s,t} := Y_s X_{s,t}$ fully determines $Z_{s,t}$. Therefore we want to find $Ξ$, such that $Z = IΞ$ is a well defined image of $Ξ$ under an abstract integration map $I$. 
Definition

We define the space $C^{\alpha, \beta}_2([0, T], W)$ of functions $\Xi$ from the simplex $0 \leq s \leq t \leq T$ into $W$ such that $\Xi_{t,t} = 0$ and

$$||\Xi||_{\alpha, \beta} := ||\Xi||_{\alpha} + ||\delta \Xi||_{\beta} < \infty,$$

where $||\Xi||_{\alpha} = \sup_{s < t} \frac{|\Xi_{s,t}|}{|t-s|^\alpha}$ as usual and also

$$\delta \Xi_{s,u,t} = \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}, \quad ||\delta \Xi||_{\beta} = \sup_{s < u < t} \frac{|\delta \Xi_{s,u,t}|}{|t-s|^\beta}.$$
Lemma 4.2 (Sewing lemma)

Let $\alpha$ and $\beta$ be such that $0 < \alpha \leq 1 < \beta$. Then, there exists a (unique) continuous map $\mathcal{I} : C^{\alpha,\beta}_2([0, T], \mathcal{W}) \to C^\alpha([0, T], \mathcal{W})$ such that $(\mathcal{I} \Xi)_0 = 0$ and

$$|(\mathcal{I} \Xi)_{s,t} - \Xi_{s,t}| \leq C|t - s|^{\beta},$$

(3)

where $C$ only depends on $\beta$ and $\|\delta \Xi\|_\beta$. 
Proof) For a partition \( \mathcal{P} \) of \([s, t]\), set

\[
\int_\mathcal{P} \Xi := \sum_{u, v \in \mathcal{P}} \Xi_{u, v}
\]

We want to show that

\[
\lim_{|\mathcal{P}| \to 0} \int_\mathcal{P} \Xi
\]

exists; this will define \( \mathcal{I} \Xi \).

Step 1. We show that given any partition \( \mathcal{P} \), we have

\[
\sup_{\mathcal{P} \subset [s, t]} \left| \Xi_{s, t} - \int_\mathcal{P} \Xi \right| \leq C \| \delta \Xi \|_\beta |t - s|^\beta.
\] (4)

Consider a partition \( \mathcal{P} \) of \([s, t]\) and let \( r \geq 1 \) be the number of intervals in \( \mathcal{P} \). When \( r \geq 2 \) there exists \( u \in [s, t] \) such that \([u_-, u], [u, u_+] \in \mathcal{P}\) for some \( u_-, u_+ \) and

\[
|u_+ - u_-| \leq \frac{2}{r - 1}|t - s|.
\]
Removing the point $u$ from $\mathcal{P}$, we obtain

$$
\left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}\setminus\{u\}} \Xi \right| = \left| \sum_{u,v \in \mathcal{P}} \Xi_{u,v} - \sum_{u,v \in \mathcal{P}\setminus\{u\}} \Xi_{u,v} \right|
$$

$$
= \left| \Xi_{u-},u + \Xi_{u,u+} - \Xi_{u-},u_+ \right|
$$

$$
= \left| \delta \Xi_{u-},u,u_+ \right|
$$

$$
\leq ||\delta F||_\beta \left( \frac{2}{r - 1} |t - s| \right)^\beta
$$
Further, we can see that if there are more than 3 elements in $\mathcal{P}$, i.e., $r \geq 3$, there exists two points $u, v' \in \mathcal{P}$ such that

$$
\left| \Xi_{s,t} - \int_{\mathcal{P}} \Xi \right| 
\leq 
\left| \Xi - \int_{\mathcal{P}\{u\}} \Xi \right| 
+ 
\left| \int_{\mathcal{P}\{u\}} \Xi - \int_{\mathcal{P}} \Xi \right| 
\leq 
\left| \Xi - \int_{\mathcal{P}\{u,v'\}} \Xi \right| 
+ 
\left| \int_{\mathcal{P}\{u,v'\}} \Xi - \int_{\mathcal{P}\{u\}} \Xi \right| 
+ 
\left| \int_{\mathcal{P}\{u\}} \Xi - \int_{\mathcal{P}} \Xi \right|
$$

By iterating this procedure, selecting a new point $u$ to remove each time, we get that the difference between $\Xi_{s,t}$ and $\int_{\mathcal{P}} \Xi$ biggest, we see that,
\[ \sup_{\mathcal{P} \subset [s,t]} \left| \Xi_{s,t} - \int_{\mathcal{P}} \Xi \right| \leq \| \delta F \|_{\beta} \sum_{i=2}^{r} \left( \frac{2}{i-1} |t - s| \right)^{\beta} \]

\[ \leq 2^{\beta} \zeta(\beta) \| \delta \Xi \|_{\beta} |t - s|^{\beta} \]

where \( \zeta(\beta) = \sum_{r=1}^{\infty} \frac{1}{r^\beta} \) is the Riemann zeta function. It then remains to show that

\[ \sup_{|\mathcal{P}| \vee |\mathcal{P}'| < \epsilon} \left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi \right| \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \]
Step 2. Take any two partitions $\mathcal{P}$ and $\mathcal{P}'$; we can suppose without loss of generality that $\mathcal{P}'$ refines $\mathcal{P}$. Then since every $[s, t] \in \mathcal{P}$ either belongs to $\mathcal{P}'$ or is divided in sub-intervals that belong to $\mathcal{P}'$, we have

$$
\left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi \right| = \left| \sum_{[u,v] \in \mathcal{P}} \left( \Xi_{u,v} - \int_{\mathcal{P}' \cap [u,v]} \Xi \right) \right| \\
\leq 2^\beta \zeta(\beta) \| \delta \Xi \|_\beta \sum_{[u,v] \in \mathcal{P}} (u - v)^\beta
$$

which proves that

$$
\sup_{|\mathcal{P}| \vee |\mathcal{P}'| < \epsilon} \left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi \right| \to 0 \text{ as } \epsilon \to 0.
$$
Therefore the limit

$$\lim_{|\mathcal{P}| \to 0} \int_{\mathcal{P}} \Xi =: (\mathcal{I} \Xi)_{s,t}$$

exists and defines a linear function $\mathcal{I}$ that, thanks to the estimate above is bounded, hence continuous.

The property $(\mathcal{I} \Xi)_0 = 0$ is trivial. Moreover, $\delta(\mathcal{I} \Xi)_{s,u,t} = 0$ for any $s, u, t \in [0, T]$. Estimate (3) follows immediately from (4).

For uniqueness, if $\mathcal{I}$ and $\tilde{\mathcal{I}}$ are two continuous linear map satisfying (3), there difference $\mathcal{I} - \tilde{\mathcal{I}}$ satisfies $(\mathcal{I} - \tilde{\mathcal{I}})_0 = 0$ and $(\mathcal{I} - \tilde{\mathcal{I}})_{s,t} \sim |t - s|^\beta$.

Since $\beta > 1$ by assumption, it follows immediately that $\mathcal{I} - \tilde{\mathcal{I}}$ vanishes identically.

$\square$
Integration of 1-form

**Theorem 4.4 (Lyons).**

Let \( \mathbf{X} = (X, \mathbb{X}) \in C^\alpha ([0, T], \mathcal{V}) \) for some \( T > 0 \) and \( \alpha > \frac{1}{3} \), and \( F : \mathcal{V} \to \mathcal{L} (\mathcal{V}, \mathbb{W}) \) be a \( C^2_b \) function. Then, the rough integral defined

\[
\int_0^1 Y_t dX_t = \lim_{|\mathcal{P}| \to 0} \sum_{[s, t] \in \mathcal{P}} \left( F(X_s)X_{s,t} + DF(X_s)\mathbb{X}_{s,t} \right)
\]

exists and one has the bound

\[
\left| \int_s^t F(X_r) d\mathbf{X}_r - F(X_s)X_{s,t} - DF(X_s)\mathbb{X}_{s,t} \right| \lesssim \|F\|_{C^2_b} \left( \|X\|_\alpha^3 + \|X\|_\alpha \|\mathbb{X}\|_{2\alpha} \right) |t - s|^{3\alpha},
\]

where the proportionality constant depends only on \( \alpha \).
Furthermore, the indefinite rough integral is $\alpha$-Hölder continuous on $[0, T]$ and we have the following quantitative estimate,

$$\| \int_0^\cdot F(X) \, dX \|_\alpha \leq C \| F \|_{C_b^2} \left( \|X\|_\alpha \vee \|X\|_{1/\alpha}^{1/\alpha} \right),$$

where the constant $C$ only depends on $T$ and $\alpha$ and can be chosen uniformly in $T \leq 1$. Furthermore, $\|X\|_\alpha = \|X\|_{\alpha} + \sqrt{\|X_2\|_{2\alpha}}$ denotes again the homogeneous $\alpha$-Hölder rough path norm.
Definition 4.6

Given a path $X \in C^\alpha([0, T], V)$, we say that $Y \in C^\alpha([0, T], \overline{W})$ is controlled by $X$ if there exists $Y' \in C^\alpha([0, T], \mathcal{L}(V, \overline{W}))$ so that the remainder term $R^Y$ given implicitly through the relation

$$Y_{s,t} = Y'_s X_{s,t} + R^Y_{s,t},$$

satisfies $||R^Y||_{2\alpha} < \infty$. This defines the space of controlled rough paths,

$$(Y, Y') \in \mathcal{D}_X^{2\alpha}([0, T], \overline{W}).$$

Although $Y'$ is not, in general, uniquely determined from $Y$ we call any such $Y'$ the Gubinelli derivative of $Y$ (with respect to $X$).
Theorem 4.10 (Gubinelli)

Let $T > 0$, $X = (X, \bar{X}) \in C^{\alpha}([0, T], V)$ for some $\alpha > \frac{1}{3}$, and $(Y, Y') \in D^{2\alpha}_X([0, T], L(V, W))$. Then there exists a constant $C$ depending only on $T$ and $\alpha$ (and $C$ can be chosen uniformly over $T \in (0, 1]$) such that

(a) The integral defined by

$$\int_0^1 YdX = \lim_{|P| \to 0} \sum_{[s, t] \in P} \left( YsX_{s,t} + Y's\bar{X}_{s,t} \right)$$

exists and, for every pair $s, t$, one has the bound

$$\left| \int_s^t Yr dX_r - YsX_{s,t} - Y's\bar{X}_{s,t} \right| \leq C \left( \|X\|_\alpha \|R^Y\|_{2\alpha} + \|\bar{X}\|_{2\alpha} \|Y'\|_\alpha \right) |t - s|^{3\alpha}.$$
(b) The map from $\mathcal{D}_X^{2\alpha}([0, T], \mathcal{L}(V, W))$ to $\mathcal{D}_X^{2\alpha}([0, T], W)$ given by

$$(Y, Y') \mapsto (Z, Z') := \left( \int_0^\cdot Y_t dX_t, Y \right),$$

is a continuous linear map between Banach spaces and one has the bound

$$||Z, Z'||_{X, 2\alpha} \leq ||Y||_{\alpha} + ||Y'||_{L^\infty} ||X||_{2\alpha} + C \left( ||X||_{\alpha} ||R^Y||_{2\alpha} + ||X||_{2\alpha} ||Y'||_{\alpha} \right).$$

* $||Y, Y'||_{X, 2\alpha} := ||Y'||_{\alpha} + ||R^Y||_{2\alpha}$
Proof) We want to apply the sewing lemma with \( \Xi_{s,t} = Y_s X_{s,t} + Y'_s \mathcal{X}_{s,t} \), therefore we need to check that \( ||\delta F||_\beta < \infty \) for some \( \beta > 1 \). By Chen’s relation,

\[
\delta F_{s,u,t} = F_{s,t} - F_{s,u} - F_{u,t} \\
= Y_s X_{s,t} - Y_s X_{s,u} - Y_u X_{u,t} + Y'_s \mathcal{X}_{s,t} - Y'_s \mathcal{X}_{s,u} - Y'_u \mathcal{X}_{u,t} \\
= - Y_{s,u} X_{u,t} + Y'_s (\mathcal{X}_{s,t} - \mathcal{X}_{s,u}) - Y'_u \mathcal{X}_{u,t} \\
= - \left( Y'_s X_{s,u} + R^Y_{s,u} \right) X_{u,t} + Y'_s (\mathcal{X}_{u,t} + X_{s,u} \otimes X_{u,t}) - Y'_u \mathcal{X}_{u,t} \\
= Y'_{u,s} \mathcal{X}_{u,t} - R^Y_{s,u} X_{u,t}.
\]
Therefore

\[ |\delta F_{s,u,t}| = |Y'_{u,s}|_\alpha |s - u|^{\alpha}|X_{u,t}|_{2\alpha}|t - u|^{2\alpha} \]
\[ + |R^Y_{s,u}|_{2\alpha}|u - s|^{2\alpha}|X_{u,t}|_\alpha|t - u|^{\alpha}, \]

hence,

\[ ||\delta F||_{3\alpha} \leq ||Y'||_\alpha ||X||_{2\alpha} + ||R^Y||_{2\alpha} ||X||_\alpha =: C_{X,Y} \]

and we can apply the sewing lemma exactly if \( \alpha > \frac{1}{3} \). The first estimate then follows immediately from step 1 in the proof of the sewing lemma. For (b), the estimate in (a) implies that

\[ |Z_t - Z_s| = \left| \int_s^t Y_r dX_r \right| \leq |Y_s X_s,t| + |Y'_s X_{s,t}| + C_{X,Y} |t - s|^{3\alpha} \]

where \( C_{X,Y} \) is the bound for \( ||\delta F|| \) above.
This can be bounded by

$$||Y||_{\infty}|X|_{\alpha}|t-s|^{\alpha} + ||Y'||_{\infty}|X|_{2\alpha}|t-s|^{2\alpha} + C_{X,Y}|t-s|^{3\alpha}$$

yielding

$$||Z||_{\alpha} \lesssim ||X||_{\alpha} (||Y||_{\alpha} + ||R^Y||_{2\alpha}) + ||X||_{2\alpha}||Y'||_{\infty},$$

thus $Z \in C^\alpha$.

By (a), $Z_{s,t} = \int_s^t Y_r \, dX_r = Y_sX_{s,t} + Y'_s \tilde{X}_{s,t} + \tilde{R}_{s,t}$ and $||\tilde{R}||_{3\alpha} < \infty$; moreover, $|Y'_s \tilde{X}_{s,t}| \leq ||Y'||_{\infty} ||\tilde{X}||_{2\alpha}|t-s|^{2\alpha}$ thus $Z \in \mathcal{D}^{2\alpha}_{\tilde{X}}$ and $Z' = Y$ (the term $Y'_s \tilde{X}_{s,t}$ is the $2\alpha$-remainder).

The last estimate for $||Z, Z'||_{\chi,2\alpha} = ||Y||_{\alpha} + ||R^Z||_{2\alpha}$ follows immediately.

Finally, continuity is given by the sewing lemma.
Thank you for your attention!