

$$\text{SU}(2) = \{ U \in \text{Mat}(2, \mathbb{C}) \mid U^\dagger U = I, \det U = 1 \}$$

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$U^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

$$U^\dagger U = I$$

$$\Rightarrow \begin{cases} |a|^2 + |c|^2 = 1 \\ \bar{a}b + \bar{c}d = 0 \\ \bar{b}a + d\bar{c} = 0 \\ |b|^2 + |d|^2 = 1 \\ ad - bc = 1 \end{cases} \Rightarrow b = -\frac{\bar{c}}{\bar{a}}d$$

$$ad - \frac{\bar{c}c}{\bar{a}}d = 0$$

$$d \left( a + \frac{bc}{\bar{a}} \right) = 1$$

$$d \frac{a\bar{a} + |c|^2}{\bar{a}} = 1 \quad d = \bar{a}$$

$$b = -\frac{\bar{c}}{\bar{a}}d = -\frac{\bar{c}}{\bar{a}}\bar{a} = -\bar{c}$$

$$U = \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix}$$

Lie algebra

$X \in \text{su}(2)$

$e^{tX} \in \text{SU}(2)$

$$\det(e^{tX}) = 1$$

$$\frac{d}{dt} (e^{tX^\dagger} e^{tX})$$

$$= \frac{d}{dt} (e^{tX^\dagger} e^{tX}) \Big|_{t=0} = (X^\dagger e^{tX^\dagger} e^{tX} + e^{tX^\dagger} X e^{tX}) \Big|_{t=0} = 0$$

$$X^\dagger = -X$$

$$\det(e^{tX}) = e^{\text{tr}(tX)} = e^{t \text{tr} X} = 1$$

$$\forall t \quad \text{tr} X = 0$$

$$X = \begin{pmatrix} a_{11} + ia_{12} & a_{21} + ia_{22} \\ a_{31} + ia_{32} & -a_{11} - ia_{12} \end{pmatrix}$$

$$X^\dagger = \begin{pmatrix} a_{11} - ia_{12} & a_{31} - ia_{32} \\ a_{21} - ia_{22} & -a_{11} + ia_{12} \end{pmatrix}$$

$$X^\dagger = -X \Rightarrow \begin{cases} a_{11} = 0 \\ a_{21} = a_{32} \\ a_{31} = a_{22} \end{cases}$$

$$[X_i, X_j] = \epsilon_{ijk} X_k$$

$$X = \begin{pmatrix} ia_{12} & a_{21} + ia_{22} \\ -a_{21} + ia_{22} & -ia_{12} \end{pmatrix}$$

$$X_1 = -\frac{i}{2} \sigma_1 = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

$$X_2 = -\frac{i}{2} \sigma_2 = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

$$X_3 = -\frac{i}{2} \sigma_3 = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}$$

$$= a_{12} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + a_{21} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$+ a_{22} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$= -2a_{12}X_1 - 2a_{21}X_2 - 2a_{22}X_3$$

$$\epsilon_{ijk} = 1 \quad | -2 \ -7 \ -7 | \quad \text{even permutation}$$

(1)

$$\epsilon_{ijk} = -1 \quad (123)(231)(321) \\ 1132) (3,21) (2,13)$$

$$SO(3) = \{ U \in M_3(\mathbb{R}) \mid U^T U = I \quad \det U = 1 \}$$

$$su(3) \quad X^T = -X \quad \text{for } X = 0$$

$$X = \begin{pmatrix} 0 & a_{11} & a_{12} \\ -a_{11} & 0 & a_{13} \\ -a_{12} & -a_{13} & 0 \end{pmatrix}$$

$$X^T = \begin{pmatrix} 0 & -a_{11} & -a_{12} \\ a_{11} & 0 & -a_{13} \\ a_{12} & a_{13} & 0 \end{pmatrix}$$

$$= a_{13} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + a_{11} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[L_1, L_2] = L_3 \quad [L_2, L_3] = L_1 \quad [L_3, L_1] = L_2$$

$$su(3) \cong so(8)$$

$$\phi: g_1 \rightarrow g_2 \quad \phi([X, Y]) = [\phi(X), \phi(Y)]$$

$$\phi(X_i) = L_i$$

$$\phi([X_i, X_j]) = \phi(\epsilon_{ijk} X_k) = \epsilon_{ijk} L_k$$

$$= [\phi(X_i), \phi(X_j)] = [L_i, L_j] = \epsilon_{ijk} L_k$$

# Quaternions

(3)

$$q = q_0 + q_1 i + q_2 j + q_3 k \quad q_i \in \mathbb{R}$$

$$\frac{q \bar{q}}{|q|^4} = 1$$

$$i^2 = j^2 = k^2 = ij = ji = -1$$

$$q^{-1} = \frac{\bar{q}}{|q|^2}$$

$$\bar{q} = q_0 - q_1 i - q_2 j - q_3 k$$

$$|q|^2 = q \bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

$$S^{3,1} = \{q \in \mathbb{H} \mid |q| = 1\} \quad \text{Spin}(3) = \{q_1 i + q_2 j + q_3 k \mid q_i \in \mathbb{R}\}$$

$$\forall t \in \mathbb{R} \quad X \in S^{3,1} \quad e^{tX} \in S^{3,1}$$

Proof:  $|e^{tX}|^2 = e^{tX} \overline{e^{tX}}$

$$\overline{e^{tX}} = \sum_{n=0}^{\infty} \frac{(\overline{tX})^n}{n!} = \sum_{n=0}^{\infty} \frac{t \overline{X}^n}{n!} = e^{t \overline{X}}$$

$$e^{tX} e^{t \overline{X}} = 1 \quad |e^{tX}|^2 = e^{t \operatorname{Re} X} = 1 \quad \forall t$$

$$X = \operatorname{Re} X + i \operatorname{Im} X$$

$$\operatorname{Re} X = 0$$

$$q = q_0 + v \quad |q|^2 = 1 \quad q_0 = \cos \theta \quad v = \sin \theta \text{ unit vector}$$

$$\bar{X} = \operatorname{Re} X - i \operatorname{Im} X$$

$$X = q_1 i + q_2 j + q_3 k$$

$$\tilde{v} = \frac{v}{|v|} \quad |\tilde{v}| = 1$$

$$\tilde{v}^2 = -1 \quad v = |v| \tilde{v} = \sin \theta \tilde{v}$$

$$\forall q \in S^{3,1} \quad \exists w \in S^{3,1} \text{ s.t. } q = e^{w}$$

$$w = \frac{q_1 i + q_2 j + q_3 k}{\sqrt{q_1^2 + q_2^2 + q_3^2}}$$

Proof:  $q \in S^{3,1} \quad q = q_0 + q_1 i + q_2 j + q_3 k$  a real part

$$v = q_1 i + q_2 j + q_3 k$$

b Im.

$$q = q_0 + v$$

$$\exists \theta \in \mathbb{R} \text{ s.t. } q_0 = \cos \theta$$

$$|v| = \sin \theta$$

$$q_0 + |v|$$

$$v = \cos \theta \tilde{v} \quad \tilde{v} \sin \theta$$

$$= e^{w}$$

(4)

$$\Phi: U(1) \rightarrow \{ \vec{v} \rightarrow U \vec{v} U^\dagger \} \in SU(2)$$

$$\Phi(U) \vec{v} = U \vec{v} U^\dagger$$

$$\Phi(U, \vec{v}) = R(U, \vec{v})$$

$$U(\theta, e_3) \in e^{iK} = \cos\theta + iK \sin\theta$$

$$(U(\theta, e_3))^\dagger = \cos\theta - iK \sin\theta$$

$$U(\theta, e_3) \vec{v} (U(\theta, e_3))^\dagger$$

$$= (\cos\theta + iK \sin\theta) (V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}) (\cos\theta - iK \sin\theta)$$

$$= (V_1 \cos^2\theta - \sin^2\theta) \hat{i} + V_2 (\sin\theta \cos\theta) \hat{j} + (2V_1 \sin\theta \cos\theta + V_3 (\cos^2\theta - \sin^2\theta)) \hat{k}$$

$$= (V_1 \cos^2\theta - V_3 \sin^2\theta) \hat{i} + (V_1 \sin\theta \cos\theta + V_3 \sin\theta \cos\theta) \hat{j} + V_3 \hat{k}$$

$$V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta - \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

homomorphism  $\forall U, V \in SU(2)$

$$\Phi(U, V) = \Phi(U, \Phi(V))$$

$$\Phi(U, V) \vec{v} = (U, V) \vec{v} (U, V)^\dagger$$

$$= U, V \vec{v} U^\dagger V^\dagger$$

$$= U (\Phi(V) \vec{v}) U^\dagger$$

$$= \Phi(U) (\Phi(V) \vec{v})$$

$$\Phi(U) = (U \vec{v} U^\dagger) = U \vec{v} U^\dagger$$

$R$  orthogonal basis

$$e_1 \in \mathbb{R} \quad e_2 \in \mathbb{J} \quad e_3 \in \mathbb{K}$$

$$W = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} \Leftrightarrow \vec{W} = W_1 \hat{i} + W_2 \hat{j} + W_3 \hat{k}$$

$$U(\theta, e_1) = e^{i\theta \hat{e}_1} = \cos\theta + i \sin\theta$$

$$U^\dagger = \cos\theta - i \sin\theta$$

$$\vec{v} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$$

$$U \vec{v} U^\dagger = (\cos\theta + i \sin\theta)$$

$$(V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}) (\cos\theta - i \sin\theta)$$

$$= V_1 \hat{i} + (V_2 \cos\theta - V_3 \sin\theta) \hat{j} + (V_2 \sin\theta + V_3 \cos\theta) \hat{k}$$

$$U(\theta, e_2)$$

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

$$U(\theta, e_2) = e^{i\theta \hat{e}_2} = \cos\theta + \mathbb{J} \sin\theta$$

$$U^\dagger = \cos\theta - \mathbb{J} \sin\theta$$

$$(\cos\theta + \mathbb{J} \sin\theta) (V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k})$$

$$= (\cos\theta + \mathbb{J} \sin\theta)$$

$$= V_1 \cos\theta + V_2 \sin\theta \hat{j} + V_3 \hat{k} + V_2 \sin\theta \hat{i} + V_3 \cos\theta \hat{j}$$

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

$$\Phi'(K) = \frac{d}{d\theta} \Phi(\cos\theta + K \sin\theta) \Big|_{\theta=0}$$

①

$$= \begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \Big|_{\theta=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2L_3$$

$$\Phi'(i) = \frac{d}{d\theta} \Phi(\cos\theta + i \sin\theta) \Big|_{\theta=0}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i \sin\theta & -\cos\theta \\ 0 & i \sin\theta & -\sin\theta \end{pmatrix} \Big|_{\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = 2L_1$$

$$\Phi'(j) = \frac{d}{d\theta} \Phi(\cos\theta + j \sin\theta) \Big|_{\theta=0}$$

$$= \begin{pmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & 0 & 0 \\ -\cos\theta & 0 & -\sin\theta \end{pmatrix} \Big|_{\theta=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = 2L_2$$

$$\Phi' \cong \mathfrak{spin} \cong \mathfrak{so}(3)$$

$$\Phi'([X, Y]) =$$

$$\Phi'([i, j]) = \Phi'(K) = 4L_3$$

$$[\Phi(i), \Phi(j)] = [2L_1, 2L_2] = 4[L_1, L_2] = 4L_3$$

$\Phi'$  real vector space

$$X = a\hat{i} + b\hat{j} + c\hat{k} \quad \Phi'(X) = 2aL_1 + 2bL_2 + 2cL_3 = 0 \Rightarrow a=b=c=0$$

$$F \text{ Spin} \rightarrow \text{Spin}$$

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$$F(q_0 + q_1 i + q_2 j + q_3 k) = \begin{pmatrix} q_0 - i q_1 & -q_2 - i q_3 \\ q_2 - i q_1 & q_0 + i q_3 \end{pmatrix}$$

$$F = (q_0 + q_1 i + q_2 j + q_3 k)$$

$$= q_0 I - q_1 \sigma_1 - q_2 \sigma_2 - q_3 \sigma_3$$

$$q \in \text{Spin} \quad |q| = 1$$

$$F(q)^T F(q) = I$$

$$q = q_0 - q_1 i - q_2 j - q_3 k$$

$$F(q) = F(q)^T = F(q^*) = (F(q))^{-1}$$

$$\det F(q) = 1$$

$$F(q) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{cases} a - i b = 1 \\ -a - i b = 0 \\ c - i d = 0 \\ c + i d = 0 \end{cases} \Rightarrow q = 1$$

$$U = \begin{pmatrix} A & -\bar{C} \\ C & \bar{A} \end{pmatrix} \quad |A|^2 + |C|^2 = 1$$

$$A = q_0 - i q_1 \quad |A|^2 + |C|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

$$C = q_2 - i q_3$$

$$i^2 = j^2 = k^2 = ijk = -1$$

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_2 = i \sigma_3$$

$$(-i \sigma_1)^2 = (-i \sigma_1) = -I$$

$$(i \sigma_1 i \sigma_2) = \sigma_2 = -i \sigma_3$$

$$\sin \theta = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \theta^{2k+1}$$

$$\cos \theta = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \theta^{2k}$$

$$-q_1 i \sigma_1$$

$$q_1 i = e^{i \theta} \text{Spin}$$

$$F(q) = e^{i \theta}$$

$$F(\exp t \tilde{v}) = \frac{d}{dt} \exp t \tilde{v}$$

$$F(\exp t \tilde{v}) = \cos |t| + \frac{\tilde{v}}{|v|} \sin |t|$$

$$F(i) F(j)$$

$$= (-i \sigma_1) (-i \sigma_2)$$

$$= -i \sigma_3$$

$$= F(k)$$

$$F(i) F(j) = F(k)$$

$$\text{injective}$$

$$F(q_1) = F(q_2)$$

$$\Rightarrow F(q_1) F(q_2)^T = I$$

$$(F(q_1))^T = F(q_2)$$

$$\Rightarrow F(q_1 q_2) = I$$

$$v_1 = q_1 q_2^{-1} \quad v_2 = \begin{pmatrix} q_0 - i q_1 & q_2 - i q_3 \\ q_2 - i q_1 & q_0 + i q_3 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} v_1 = 0 & v_2 = 1 \\ v_2 = 0 & v_1 = 1 \\ v_3 = 0 & v_4 = 1 \end{matrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

# Representation of $SU(2)$ and $SO(3)$

①

$$U(1) = \{ e^{i\theta} \mid \theta \in \mathbb{R} \}$$

$U(1)$  is commutative. Schur's Lemma guarantees all of irreducible representations are one-dimensional.

$$\rho_K(e^{i\theta}) = e^{iK\theta}$$

$$\Phi: SU(2) \rightarrow SO(3)$$

$$g = \begin{pmatrix} a & b \\ -\bar{b} & a \end{pmatrix} \quad |a|^2 + |b|^2 = 1 \quad a, b \in \mathbb{C}$$

$$\Phi(g)X = gXg^{-1}$$

$$f: f^{-1}SU(2) \rightarrow SU(2)$$

$$F: Sp(1) \rightarrow SU(2)$$

$$F(q) = \begin{pmatrix} q_0 - iq_3 & -q_1 - iq_2 \\ q_1 - iq_2 & q_0 + iq_3 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\Phi: Sp(1) \rightarrow SO(3)$$

$$\Phi_{SU(2)} = \Phi_{Sp(1)} \circ f$$

$$\Phi_{Sp(1)}(q) = q \sqrt{q}^{-1}$$

$$= \Phi_{Sp(1)}(f(q))$$

example:  $g \in SU(2) = \begin{pmatrix} e^{i\frac{\pi}{4}} & 0 \\ 0 & -\frac{i}{4} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \end{pmatrix}$

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

$$g = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} a & b \\ \bar{b} & a \end{pmatrix} \quad a=i, b=0$$

$$\begin{cases} q_0 - iq_3 = i \\ q_1 + iq_2 = 0 \end{cases} \Rightarrow \begin{cases} q_0 = 1 \\ q_3 = -1 \end{cases}$$

$$q = q_0 + q_1 i + q_2 j + q_3 k = -k = f(q)$$

$$\Phi_{\text{spin}}(x)\vec{v} = (K|\vec{v}K$$

(2)

$$① \vec{v} = i \quad (-K|iK = -iK = -\cancel{i}K = -i$$

$$② \vec{v} = j \quad (-K|jK = -(-i)K = -j$$

$$③ \vec{v} = K \quad (-K|KK = (-1)K = K$$

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\pi_0 \quad \text{SU}(2) \rightarrow \text{GL}(C)$$

$$\pi_1(g) = g \quad \text{Spin } \frac{1}{2}$$

$$\rho_0 \quad \text{SO}(3) \rightarrow \text{GL}(C)$$

$$\pi_1, g$$

$$\pi_1 \quad \text{SU}(2) \rightarrow \text{GL}(C^3)$$

$$\rho_1 \quad \text{SO}(3) \rightarrow \text{GL}(C^3)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sin \theta = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \theta^{2k+1}$$

$$\cos \theta = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \theta^{2k}$$

$$\sigma^2 = \sigma_3 \cdot \sigma_3 = 1$$

$$e^{i\theta \sigma_3} = \cos \theta + i \sin \theta \sigma_3 = \begin{pmatrix} \cos \theta + i \sin \theta & 0 \\ 0 & \cos \theta - i \sin \theta \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$e^{i\theta \sigma_1} = \cos \theta + i \sin \theta \sigma_1 = \begin{pmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix}$$

$$e^{i\theta \sigma_2} = \cos \theta + i \sin \theta \sigma_2 = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$$\sigma_1^n = \begin{cases} I & n \text{ even} \\ \sigma_1 & n \text{ odd} \end{cases}$$

$$\sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} \sigma_1^k = \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} I + \sum_{k=0}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!} \sigma_1$$

$$= \cos \theta$$

Th: If  $q$  in the set  $\{q_j\}$ , so is  $-q$ .

(3)

Pr:  $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$   $P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$P \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} P^{-1} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad P g(\theta) P^{-1} = g(-\theta)$$

$$\pi(AB A^{-1}) = \pi(A) \pi(B) \pi(A^{-1}) \\ = \pi(A) \pi(B) \pi(A)^{-1}$$

$$\pi(A A^{-1}) = \pi(A) \pi(A^{-1}) = I \quad \pi(A^{-1}) = \pi(A)^{-1}$$

$$\pi(P g(\theta) P^{-1}) = \pi(g(-\theta))$$

$$\pi(g(\theta)) = \begin{pmatrix} e^{i q_1 \theta} & & \\ & \ddots & \\ & & e^{i q_n \theta} \end{pmatrix}$$

$$\pi(g(\theta)) V_{q_1} = e^{-i q_1 \theta} V_{q_1}$$

$$W = \pi(P) V_{q_1}$$

$$\pi(g(\theta)) W = \pi(g(\theta)) \pi(P) V_{q_1} = \pi(P) (\pi(P)^{-1} \pi(g(\theta)) \pi(P)) V_{q_1}$$

$$= \pi(P) (\pi(g(-\theta)) V_{q_1})$$

$$= \pi(P) (e^{-i q_1 \theta} V_{q_1})$$

$$= e^{-i q_1 \theta} \pi(P) V_{q_1} = e^{-i q_1 \theta} W$$

$V_{q_1}$  is the eigenvector of  $q_1$

$W = \pi(P) V_{q_1}$  is the eigenvector of  $-q_1$

(4)

$$\Phi(-u)(\vec{v}) = (-u)\vec{v}(-u)^T \\ = (-u)\vec{v} - u^T = \Phi(u)\vec{v}$$

$$U = \cos\theta + \vec{v}\sin\theta$$

$$U(\theta, e_3) = \cos\theta + K\sin\theta$$

$$\Phi(U) = R(2\theta, e_3) = \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$-U = -\cos\theta - K\sin\theta = U(\theta + \pi, e_3)$$

$$\Phi(-U) = R(U(\theta + \pi), e_3) = R(U(\theta), e_3)$$

$$q = a + bi + c\vec{j} + d\vec{k} \quad a, b, c, d \in \mathbb{R}$$

$$|q|^2 = 1$$

$$\forall U \in \text{Spin}(1) \quad U = e^X \quad X \text{ is pure imaginary quaternions}$$

$$U = a + \vec{v}$$

$$\text{Case 1: } \vec{v} = 0 \quad U = a \quad U = 1 \text{ or } -1 \quad X = 0$$

$$\vec{v} \neq 0 \quad \theta = \arccos a \quad 0 < \theta < \pi$$

$$|\vec{v}| = \sin\theta > 0$$

$$W = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}}{\sin\theta} \quad \vec{v} = \sin\theta W$$

$$U = a + \vec{v} = \cos\theta + \vec{v}/\sin\theta \sin\theta$$

the lie algebra of  $su(2)$

(5)

$$X_1 = -\frac{i}{2}\sigma_1 = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \quad X_2 = -\frac{i}{2}\sigma_2 = -\frac{i}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad X_3 = -\frac{i}{2}\sigma_3 = -\frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}$$

$$su(2) = \{X \in M_2(\mathbb{C}) \mid X^\dagger = -X, \text{ or } X=0\}$$

$$X_1^\dagger = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} = -X_1$$

$$X = \begin{pmatrix} a_1 + ia_2 & a_3 + ia_4 \\ a_3 + ia_4 & a_1 + ia_2 \end{pmatrix}$$

$$X^\dagger = -X \quad a_1 = a_4 = 0$$

$$\overline{a_3 + ia_4} = a_3 - ia_4 = -(a_3 + ia_4)$$

$$a_3 = -a_4$$

$$a_3 = a_4$$

$$X = \begin{pmatrix} ia_2 & a_3 + ia_4 \\ -a_3 + ia_4 & -ia_2 \end{pmatrix}$$

$$= a_2 \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + a_4 \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}$$

$$[X_1, X_2] = X_3 \quad [X_2, X_3] = X_1 \quad [X_3, X_1] = X_2$$

$$S_j = iX_j = \frac{1}{2}\sigma_j \quad S_1 = \frac{1}{2}\sigma_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad S_2 = \frac{1}{2}\sigma_2 = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \quad S_3 = \frac{1}{2}\sigma_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$\pi: \mathrm{SU}(2) \rightarrow \mathrm{U}(1) \quad \pi': \mathrm{SU}(1|1) \rightarrow \mathrm{U}(1)$$

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$$\pi(g) = g$$

$$\pi'(X) = X$$

$$S_3 = \frac{1}{2} \sigma_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$\mathrm{SU}(2) \quad X^\dagger = -X \quad \text{tr} X = 0$$

$$X = 2iS_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$X^2 S_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad X^\dagger = -X \quad \text{tr} X = 0$$

$$e^{\theta 2iS_3} = \begin{pmatrix} e^{\theta i} & 0 \\ 0 & e^{-\theta i} \end{pmatrix}$$

$$iX = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (iX)^\dagger = -iX$$

$$\pi'_C: \mathrm{SU}(1|1)_C \rightarrow \mathrm{U}(1)_C$$

$$\forall Z \in \mathrm{SU}(1|1)_C \quad Z = X + iY \quad X, Y \in \mathrm{SU}(1|1)$$

$$W = A + iB \quad A, B \in \mathrm{U}(1)$$

$$\pi'_C(X + iY) = \pi'_C(X) + i\pi'_C(Y)$$

$$X = 2iS_3 \in \mathrm{SU}(1|1) \quad S_3 = -\frac{1}{2i} X$$

$$\pi'_C(2iS_3) = 2i\pi'_C(S_3)$$

$$\pi'_C(S_3) = \frac{1}{2i} \pi'_C(2iS_3) = \frac{1}{2i} \pi'_C(2iS_3) = \begin{pmatrix} \frac{q_1}{2} & & \\ & \ddots & \\ & & \frac{q_m}{2} \end{pmatrix}$$

$$V_K = \{ v \in V \mid \pi'_C(S_3)v = \frac{K}{2}v \}$$

$$\pi'_C(S_3) \pi'_C(S_1)v = (\pi'_C(S_3) \pi'_C(S_1))v$$

$$= (\pi'_C(S_1) \pi'_C(S_3) + [\pi'_C(S_3), \pi'_C(S_1)])v$$

$$= \pi'_C(S_1) \pi'_C(S_3)v + \pi'_C(iS_2)v$$

$$= \pi'_C(S_1) \frac{K}{2}v + \pi'_C(iS_2)v$$

$$= \frac{K}{2} (\pi'_C(S_1)v) + \pi'_C(iS_2)v$$

Define raising and lowering operators

$$S_+ = S_1 + iS_2, \quad S_- = S_1 - iS_2$$

$$[S_3, S_+] = S_+ \quad [S_3, S_-] = -S_-$$

$$\pi(S_3) \pi(S_+)V = \pi(S_+) (\pi(S_3)V) = \pi(S_+)V$$

$$\pi(S_3) \pi(S_+)V = \left(\frac{K+1}{2}\right) \pi(S_+)V$$

$$\pi(S_3) \pi(S_-)V = \left(\frac{K-1}{2}\right) \pi(S_-)V$$

Th: Finite dimensional irreducible representations of  $su(2)$  have weights of the form

$$-n, -n+2, \dots, n-2, n$$

for a non-negative integer, each with multiplicity  $n$ , with  $n$  a highest weight

$$\text{for } K \quad \pi(S_3)V = \frac{K}{2}V$$

$$n=0 \quad \pi(S_3)V = \frac{K}{2}V = 0$$

$$\pi(S_3)W_K = \pi(S_3)(\pi(S_-)^K V) = (j-K)V \in W$$

$$\pi(S_-)W_K = \pi(S_-)(\pi(S_-)^K V) = \pi(S_-)^{K+1} V = W_{K+1} \quad \text{GW}$$

$$W = \{W_0, W_1, \dots, W_m\}$$

$$W_K = (\pi(S_-))^K V \quad W = \text{span}\{W_0, \dots, W_m\}$$

$$W_m \neq 0 \quad \pi(S_-)W_m = W_{m+1} = 0$$

$$\pi(S_+) \pi(S_-) = \pi(S_-) \pi(S_+) + 2\pi(S_3)$$

$$\pi(S_+)W_K \in W$$

$$\pi(S_+)W_{K+1} = \pi(S_+)(\pi(S_+)W_K + 2\pi(S_3)W_K)$$

$$\pi(g)f(x) = f(g^{-1}(x))$$

$$(\pi(g_2 g_1)f)(x) = f((g_2 g_1)^{-1}x) = f(g_1^{-1} g_2^{-1} x)$$

$$\pi(g_1)\pi(g_2)f = \pi(g_1)f(g_2^{-1}x) = f(g_1^{-1} g_2^{-1} x) = f(g_2^{-1} g_1^{-1} x)$$

$$\pi(g_2 g_1) = \pi(g_1) \circ \pi(g_2)$$

Homogeneous polynomials

$$f(z, \bar{z}) = a_0 z^n + a_1 z^{n-1} \bar{z} + \dots + a_m z \bar{z}^m + a_n \bar{z}^n$$

$$g \cdot v = g(z_1 e_1 + z_2 e_2)$$

$$v' = g v \quad v = z_1' e_1 + z_2' e_2$$

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = g \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\pi(g)f(v) = f(g^{-1}v)$$

$$\text{dual basis } \{z_i, \bar{z}_i\} \in V^* \quad z_i(e_j) = \delta_{ij}$$

$$v = a e_1 + b e_2 \in V$$

$$z_1(v) = a \quad z_2(v) = b$$

$$\phi \in V^* \quad (g \cdot \phi)(v) = \phi(g^{-1}v)$$

$$g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$g^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} (z, \bar{z})$$

$$v = \begin{pmatrix} z_1(v) \\ z_2(v) \end{pmatrix}$$

$$(g \cdot z_1)(v) = z_1(g^{-1}v)$$

$$(g \cdot z_2)(v) = z_2(g^{-1}v)$$

$$z_i(g^{-1}v) =$$

$$g^{-1}(v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1(v) \\ z_2(v) \end{pmatrix}$$

$$g^{-1}(v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1(v) \\ z_2(v) \end{pmatrix}$$

$$= (a(z_1(v)) + b(z_2(v))) e_1 + (c(z_1(v)) + d(z_2(v))) e_2$$

$$z_1(g^{-1}v) = a z_1(v) + b z_2(v)$$

$$(g \cdot z_1)(v) = (a z_1 + b z_2)(v)$$

~~g~~

dual basis

$$z_1, z_2 \in V^*$$

$$\begin{cases} z_1(e_1) = 1 & z_1(e_2) = 0 \\ z_2(e_1) = 0 & z_2(e_2) = 1 \end{cases}$$

$$(g\phi)(x) = \phi(g^T x)$$

$$g^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\forall x \in V$$

$$x = z_1(x)e_1 + z_2(x)e_2$$

$$g^T x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$$

$$= (az_1(x) + bz_2(x))e_1 + (cz_1(x) + dz_2(x))e_2$$

$$z_1(g^T x) = az_1(x) + bz_2(x)$$

$$g \cdot z_1(x) = (az_1 + bz_2)(x)$$

$$\begin{pmatrix} g \cdot z_1 \\ g \cdot z_2 \end{pmatrix} = g^T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$(g \cdot \phi)(aV + bV_2)$$

$$= a(g \cdot \phi)(V_1) + b(g \cdot \phi)(V_2)$$

$$(g \cdot \phi)(aV_1 + bV_2) = \phi(g^T(aV_1 + bV_2))$$

$$= \phi(ag^T V_1 + bg^T V_2)$$

$$= a\phi(g^T V_1) + b\phi(g^T V_2)$$

$$= a(g \cdot \phi)(V_1) + b(g \cdot \phi)(V_2)$$

$$\pi_n(g)(f) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = f \left( g^T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)$$

$$g(t) = e^{tX}$$

$$\pi'_n(X)f = \frac{d}{dt} \Big|_{t=0} \pi_n(e^{tX})f$$

$$\pi_n(e^{tX})f = f(e^{tX} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = f(e^{tX} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = \begin{pmatrix} -\lambda_1 z_1 + \lambda_2 z_2 \\ -\lambda_2 z_1 + \lambda_1 z_2 \end{pmatrix}$$

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$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = e^{-tX} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\frac{d}{dt} f(z_1(t), z_2(t)) = \frac{\partial f}{\partial z_1} \frac{dz_1}{dt} + \frac{\partial f}{\partial z_2} \frac{dz_2}{dt}$$

$$\frac{d}{dt} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \Big|_{t=0} = -X e^{-tX} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -X \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= - \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

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$$X_0 = -\frac{i\partial}{\partial z} \quad X_3 = -\frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X_{11} = -\frac{i}{2} \quad X_{12} = X_{21} = 0 \quad X_{22} = \frac{i}{2}$$

$$\begin{aligned} \pi'_n(s_2)f &= -X_1 z_1 + X_2 z_2 \frac{\partial f}{\partial z_1} - (X_1 z_1 + X_2 z_2) \frac{\partial f}{\partial z_2} \\ &= -\left(-\frac{i}{2} z_1\right) \frac{\partial f}{\partial z_1} - \left(\frac{i}{2} z_2\right) \frac{\partial f}{\partial z_2} \\ &= \frac{i z_1}{2} \left( \frac{\partial f}{\partial z_1} - z_2 \frac{\partial f}{\partial z_2} \right) \end{aligned}$$

$$f = z_1^{nk} z_2^k$$

$$\frac{\partial f}{\partial z_1} = nk z_1^{nk-1} z_2^k$$

$$\frac{\partial f}{\partial z_2} = k z_1^{nk} z_2^{k-1}$$

$$\pi'_n(s_2)f = \frac{i}{2} nk f - \frac{i}{2} k f = \frac{i}{2} (z_1 \frac{\partial f}{\partial z_1} - z_2 \frac{\partial f}{\partial z_2})$$

$$X_1 = -\frac{i}{2} \partial_1 = -\frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad X_2 = -\frac{i\partial}{\partial z} = -\frac{i}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

$$X_3 = -\frac{i\partial}{\partial z} = -\frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\pi'_n(s_1) z_1^k z_2^{nk} = \frac{1}{2} (-k z_1^k z_2^{nk} + (nk) z_1^k z_2^{nk}) = \frac{1}{2} (nk) z_1^k z_2^{nk}$$

$$S_1 = S_1 + i S_2$$

$$\pi'_n(s_1) \pi'_n(s_2) \pi'_n(s_1) = -z_2 \frac{\partial}{\partial z_1}$$

$$\pi'_n(s_1) \pi'_n(s_2) z_1^n = z_2 \frac{\partial}{\partial z_1} (z_1^n) = 0$$

$$\begin{aligned} \pi'_n(s_3) z_1^n &= -\frac{1}{2} (z_1 \frac{\partial}{\partial z_1} z_1^n - z_2 \frac{\partial}{\partial z_2} z_1^n) \\ &= -\frac{1}{2} \frac{n}{z_1} z_1^n \end{aligned}$$

$\pi$  irreducible  $\Leftrightarrow \pi'$  irreducible

(1)

Proof: Assume that  $\pi$  reducible

$\pi \hookrightarrow \pi|_W$

$\exists w \in V$  s.t.  $w \neq 0$  and  $w \neq \checkmark$  s.t.  $\forall g \in G \pi(g)w \in W$

$\forall x \in g \pi'(x)w \in W$

$\forall w \in W \forall x \in g$

$g_t = e^{tx} \in G \quad \pi(g_t)w \in W$

$$\pi'(x)w = \lim_{t \rightarrow 0} \frac{\pi(e^{tx})w - w}{t}$$

$$\pi'(x)w = \frac{d}{dt} \pi(e^{tx})w \Big|_{t=0} = \lim_{t \rightarrow 0} \frac{\pi(e^{tx})w - w}{t} \in W$$

Set  $f = g = z_1$   $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$

$$\langle f, g \rangle = \int_{\mathbb{C}} |z_1|^2 dx_1 dy_1 \int_{\mathbb{C}} dx_2 dy_2$$

$$x_1 = r_1 \cos \theta_1, y_1 = r_1 \sin \theta_1$$

$$= \int_0^\infty \int_0^{2\pi} r_1^2 \int_0^\infty \int_0^{2\pi} r_1 r_2 d\theta_1 d\theta_2 dr_1 dr_2$$

$$I = \int_{\mathbb{C}^2} |z_1|^2 e^{-|z_1|^2} |z_1|^2 dx_1 dy_1 dx_2 dy_2$$

$$= \int_{\mathbb{C}} |z_1|^2 e^{-|z_1|^2} dx_1 dy_1 \cdot \int_{\mathbb{C}} e^{-|z_1|^2} dx_2 dy_2$$

$$= \int_0^\infty \int_0^{2\pi} r^2 e^{-r^2} r dr d\theta \int_0^\infty \int_0^{2\pi} e^{-r^2} r dr d\theta$$

$$\int_0^{\infty} e^{-r} r dr = \frac{1}{2} \int_0^{\infty} e^{-u} du = \frac{1}{2}$$

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$$\int_0^{2\pi} \int_0^{\infty} r^2 e^{-r^2} r dr d\theta = 2\pi \int_0^{\infty} r^3 e^{-r^2} dr$$

$$= \frac{1}{2} \int_0^{\infty} u e^{-u} du$$

$$= \Gamma(2) = 1$$

$$f = z_1^j \bar{z}_2^k \quad g = z_1^{j'} \bar{z}_2^{k'}$$

$$\langle f, g \rangle = \frac{1}{\pi^2} \int_C \bar{z}_1^j z_1^{j'} e^{-|z|^2} dx dy,$$

$$\int_C \bar{z}_2^k z_2^{k'} e^{-|z|^2} dx dy,$$

$$z = r e^{i\theta} \quad \bar{z} = r e^{-i\theta} \quad dx dy = r dr d\theta$$

$$\int_0^{2\pi} e^{i\theta m} d\theta = \begin{cases} 2\pi & m=0 \\ 0 & m \neq 0 \end{cases}$$

$$m = j - j'$$

$$A(j, j') = \int_0^{\infty} \int_0^{2\pi} r (e^{-i\theta})^j (r e^{i\theta})^{j'} e^{-r^2} r dr d\theta$$

$$= \int_0^{\infty} r^{j+j'+1} e^{-r^2} dr \int_0^{2\pi} e^{i\theta(j-j')} d\theta$$

$$j=j' \quad A(j, j) = 2\pi \int_0^{\infty} r^{2j+1} e^{-r^2} dr$$

$$= \frac{1}{2} \int_0^{\infty} u^j e^{-u} du = \frac{1}{2} \Gamma(j+1) = \frac{1}{2} j!$$

$$A(j, j) = 2\pi \frac{1}{2} j! = \pi j! \quad B(K, K') = \begin{cases} \pi K! & K=K' \\ 0 & K \neq K' \end{cases}$$

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$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = g^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} z_1' = Az_1 + Bz_2 \\ z_2' = Cz_1 + Dz_2 \end{cases}$$

$$(\pi_n g) f(z_1, z_2) = f(z_1', z_2')$$

$$f(z_1, z_2) = z_1^j z_2^{n-j}$$

$$\pi_n(g) f = (z_1')^j (z_2')^{n-j}$$

$$= (Az_1 + Bz_2)^j (Cz_1 + Dz_2)^{n-j} \in V_n$$

$$\forall f = \sum_{j=0}^n g_j z_1^j z_2^{n-j}$$

$$\pi_n(g) f = \sum_{j=0}^n g_j \pi_n(g) (z_1^j z_2^{n-j}) \in V_n$$

$$n=2 \quad \begin{cases} z_1' = \alpha_1 z_1 + \beta_1 z_2 \\ z_2' = \alpha_2 z_1 + \beta_2 z_2 \end{cases}$$

$$f = z_2^2 \quad \begin{cases} z_1' = \alpha_1 z_1 + \beta_1 z_2 \\ z_2' = \alpha_2 z_1 + \beta_2 z_2 \end{cases}$$

$$\pi_2(g) z_2^2 = (z_2')^2 = (\alpha_2 z_1 + \beta_2 z_2)^2$$