

Chapter I: Introduction and Overview

(States)

Axiom 1: The state of a quantum mechanical system is given by a non-zero vector in a complex vector space H with Hermitian inner product $\langle \cdot, \cdot \rangle$.

$H = \mathbb{C}^2, |\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ or $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$.

If we have two states, $|\psi\rangle, |\phi\rangle$ $\langle \phi | \psi \rangle = \phi_1^* \psi_1 + \phi_2^* \psi_2 + \dots + \phi_n^* \psi_n$.

$\langle \psi | = |\psi\rangle^*$ +: Hermitian conjugate.

$|\phi\rangle = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ $|\psi\rangle = \begin{pmatrix} 2 \\ 3i \end{pmatrix}$

$\langle \phi | = (1, -i)$ $\langle \phi | \psi \rangle = (1, -i) \begin{pmatrix} 2 \\ 3i \end{pmatrix} = 2 + 3 = 5$

two differences: The state space is always linear.

The state space is a complex vector space

(Quantum observables)

Axiom 2: The observables of a quantum mechanical system are given by self-adjoint linear operators on H .

(Dynamics)

Axiom 3: There is a distinguished quantum observable, the Hamiltonian H . The time evolution of states $|\psi(t)\rangle \in H$ is given by the Schrödinger equation:

$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$

$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle$

(1.1)

The operator has eigenvalues that are bounded below

$\hbar = 1$. for simplifying simplifying

$\hbar \rightarrow 0$. "classical limit"

Principle: State for which the value of an observable. ⑤
 can be characterized by a well-defined number are the states
 that are eigenvectors for the corresponding self-adjoint operator.
 The value of the observable in such a state will be a
 real number. the eigenvalue of the operator.

$$\langle \psi | \hat{A} | \psi \rangle = \lambda \langle \psi | \psi \rangle = (\langle \psi | \hat{A} | \psi \rangle)^* \Rightarrow \lambda \text{ is a real number.}$$

principle: (The Born rule). Observable $\hat{O}$ and two unit-norm
 States $|\psi_1\rangle$ and $|\psi_2\rangle$ that are eigenvectors of $\hat{O}$
 with distinct eigenvalues λ_1 and λ_2

$$\hat{O} |\psi_1\rangle = \lambda_1 |\psi_1\rangle, \quad \hat{O} |\psi_2\rangle = \lambda_2 |\psi_2\rangle$$

the complex linear combination state

$$c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$$

will ~~not~~ have a well-defined value for the observable $\hat{O}$.

If one attempts to measure this observable, one will
 get either λ_1 or λ_2 , with probabilities

$$\frac{|c_1|^2}{|c_1|^2 + |c_2|^2} \quad \text{and} \quad \frac{|c_2|^2}{|c_1|^2 + |c_2|^2} \quad \text{respectively.}$$

Definition: Group action on a set).

$$(g, x) \in G \times M \rightarrow g \cdot x \in M.$$

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \quad (1.2)$$

and $e \cdot x = x$ where e is the identity element of G .

We study $F(M)$, the set of functions on M , instead of studying M . A action g of G act on some function f on M (1.31).

$$(g \cdot f)(x) = f(g^{-1} \cdot x).$$

This implies

$$g_1 \cdot (g_2 \cdot f)(x) = (g_2 \cdot f)(g_1^{-1} \cdot x) = f(g_2^{-1} \cdot (g_1^{-1} \cdot x))$$

The group action property is ~~not~~ ^{guaranteed}.

$$= f((g_2^{-1} g_1^{-1}) \cdot x)$$

$$= f((g_1 g_2)^{-1} \cdot x)$$

$$= (g_1 g_2) \cdot f$$

With this preparation, we define "representation" as an action of a group by linear transformations on a vector space:

Definition: A representation (π, V) of a group is a homomorphism $\pi: G \rightarrow GL(V)$ ($\pi(g_1 g_2) = \pi(g_1) \pi(g_2)$).

$$\pi: g \in G \mapsto \pi(g) \in GL(V).$$

where $GL(V)$ is the group of invertible linear maps $V \rightarrow V$, with V a vector space.

Example: • $M = \{x_1, x_2, x_3\}$, $F(M) = \mathbb{C}^3$. For $f \in F(M)$, f is a vector in $\mathbb{C}^3$, with components $(f(x_1), f(x_2), f(x_3))$.

• Take $G = S_3$ the group of permutations of 3 elements.

This group has $3! = 6$ elements

• Take G to act on M by permuting the 3 elements. ④

$$(g, x_j) \mapsto g \cdot x_j$$

$$g = (321) \quad \overbrace{3 \rightarrow 2 \rightarrow 1}$$

• This group action provides a representation of G on Fun by the linear $(\pi(g)f)(x_j) = f(g^{-1} \cdot x_j)$. $g^{-1} = (312)$

$$\pi(g) \begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{pmatrix} = \begin{pmatrix} f(x_2) \\ f(x_3) \\ f(x_1) \end{pmatrix} \Rightarrow \pi(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Verify:

$$\boxed{\pi(g_1) \pi(g_2) = \pi(g_1 g_2)}$$

special case
for permutation
representation.

(π, V) π is also very important.

$\pi = I_3, \forall g \in G$ trivial representation.

Next, we see which unitary group representation is important.

Definition (Unitary representation) A representation (π, V) on a complex vector space V with Hermitian inner product $\langle \cdot, \cdot \rangle$ is a unitary representation if it preserves the inner product, i.e.,

$$\langle \pi(g)v_1, \pi(g)v_2 \rangle = \langle v_1, v_2 \rangle$$

for all $g \in G$ and $v_1, v_2 \in V$.

Why unitary is important?

$$U^{-1} = U^\dagger$$

$U^\dagger$ (dagger)

physics.
math "*"

$$G = (\mathbb{R}, +)$$

one-parameter unitary group

$$t \in \mathbb{R} \rightarrow \lambda(t) = e^{-\frac{i}{\hbar} H t}$$

Unitary group.
 $\lambda(0) = I$
 $\lambda(t_1 + t_2) = \lambda(t_1) \lambda(t_2)$

$$\left. \frac{d}{dt} \lambda(t) \right|_{t=0} = -\frac{iH}{\hbar} =: A$$

$$A^\dagger = -A$$

(skew-adjoint)

Physics:

$$H = \hbar A$$

$$H^\dagger = H$$

(Hamiltonian)

$$\lambda(t)^\dagger \lambda(t) = I$$

$$\lambda(t) = I + tA + o(t^2)$$

$$A = \left. \frac{d\lambda}{dt} \right|_{t=0}$$

$$\lambda(t)^\dagger = I + tA^\dagger + o(t^2)$$

$$\lambda(t)^\dagger \lambda(t) = I = I + t(A^\dagger + A) + o(t^2)$$

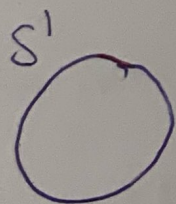
$$A^\dagger + A = 0 \quad A^\dagger = -A$$

Chapter 2: The Group $U(1)$ and its representations

$$U(1) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$$

$$|e^{i\theta}| = 1$$

$$e^{i\theta_1 + i\theta_2} = e^{i\theta_1} e^{i\theta_2}$$



$$\theta \sim \theta + 2\pi$$

$$U U^\dagger e^{i\theta} e^{-i\theta} = I$$

Continuous

discrete

Lie group

$$U(1) \Rightarrow \underline{q \in \mathbb{Z}} \Rightarrow \mathcal{Q} \Rightarrow [H, \mathcal{Q}] = 0$$

quantization

observable

$\mathcal{Q}$ conserved

Let us first begin with the group $U(1)$.

Definition (The group $U(1)$). The elements of the group $U(1)$ are points on the unit circle, which can be labeled by a unit complex number $e^{i\theta}$, or an angle $\theta \in \mathbb{R}$ with θ and $\theta + 2\pi$ labeling the

Same group element for $N \in \mathbb{Z}$. Multiplication of group elements is complex multiplication, which by the properties of the exponential satisfies

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$$

So in terms of angles the group law is addition (mod 2π)

To better understand the structure of $U(1)$, we introduce "irreducible representation".

Definition. A representation π is called irreducible if it has no subrepresentation, meaning non-zero proper subspaces $W \subset V$ such that $(\pi|_W, W)$ is a representation. A representation that does have such a subrepresentation is called reducible.

Definition (direct sum representation)

$$\dim(\pi_1) = n_1, \dim(\pi_2) = n_2, \dim(\pi_1 \oplus \pi_2) = n_1 + n_2$$

$$(\pi_1 \oplus \pi_2): g \in G \rightarrow \begin{pmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{pmatrix}$$

For unitary representations

Theorem 2.1. Any unitary representation π can be written as a direct sum $\pi = \pi_1 \oplus \pi_2 \oplus \dots \oplus \pi_m$.

where the π_i are irreducible.

proof. If (π, V) is not irreducible there exists a non-zero $W \subset V$ such that $(\pi|_W, W)$ is a representation

and $(\pi, V) = (\pi|_W, W) \oplus (\pi|_{W^\perp}, W^\perp)$.

W is invariant subspace. Since $\pi(g)W \subseteq W, \forall g \in G$.

⑨
 V has Hermitian inner product.

We define $W^\perp = \{v \in V : \langle v, w \rangle = 0, \forall w \in W\}$ $V = W \oplus W^\perp$

Next, we verify $W^\perp$ is invariant, i.e.,

$$\forall v \in W^\perp \Rightarrow \pi(g)v \in W^\perp \Rightarrow \langle \pi(g)v, w \rangle = 0, \forall w \in W.$$

Since W is invariant. $g^{-1} \in G \Rightarrow \pi(g^{-1})w \in W$.

$$\langle \pi(g)v, w \rangle = \langle \pi(g)v, \pi(g)\pi(g^{-1})w \rangle \stackrel{\text{unitary}}{=} \underbrace{\langle v, \pi(g^{-1})w \rangle}_{w \in W} = 0.$$

$$\Rightarrow \pi(g)W^\perp \subseteq W^\perp.$$

$$\Rightarrow (\pi, V) = (\pi|_W, W) \oplus (\pi|_{W^\perp}, W^\perp).$$

$\Rightarrow V$ is finite. $\cong \mathbb{C}^n \Rightarrow$ By induction method, the proof is finished.

Finding the decomposition of an arbitrary unitary representation into irreducible components can be a very non-trivial problem.

Theorem (Schur's lemma). If a complex representation (π, V) is irreducible, then the only linear maps $M: V \rightarrow V$ commuting with all the $\pi(g)$ are λI .

Proof: Assume that M commutes with all the $\pi(g)$. We want to show that (π, V) irreducible implies $M = \lambda I$.

$(M = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix})$ $\det(M - \lambda I) = 0$ must have solution by Fundamental Theorem of Algebra.

The eigenspaces $V_\lambda = \{v \in V : Mv = \lambda v\} =: \ker(M - \lambda I) \neq \{0\}$.

Since this operator and all the $\pi(g)$ commute, we have

$$v \in \ker(M - \lambda I) \Rightarrow \pi(g)v \in \ker(M - \lambda I).$$

$$M \pi(g)v \stackrel{\text{commute}}{=} \pi(g)Mv = \pi(g)\lambda v = \lambda \pi(g)v$$

Therefore, $\ker(M - \lambda I) \subseteq V$ is a representation of G . $\square$

If V is irreducible, we must have either $\ker(M - \lambda I) = V$ or $\ker(M - \lambda I) = 0$. Since λ is an eigenvalue, $\ker(M - \lambda I) \neq 0$.
 $\Rightarrow \ker(M - \lambda I) = V \Rightarrow M = \lambda I$. $\square$

$\Rightarrow$ Theorem 2.2. If G is commutative, all of its irreducible representations are one dimensional.

proof. For G commutative, $g \in G$, any representation will satisfy
 $\pi(gh) = \pi(g)\pi(h) = \pi(h)\pi(g) = \pi(hg)$ (since $gh = hg$).

If π is irreducible, since $\pi(h)$ commutes with all the $\pi(g)$.

We use Schur's lemma to see. $\pi(h) = \lambda_h I$

Suppose $\dim V > 1$ take $v \in V$. $W = \text{span}\{v\}$
 $\pi(h)W = \lambda_h W = \lambda_h I W \Rightarrow \pi(h)v \in W \Rightarrow W$ is invariant.

But if $\dim V > 1$. $0 \neq W \neq V \Rightarrow V$ is reducible.

Therefore $\dim V = 1$ $\square$

By theorem 2.2, since $U(1)$ is a commutative group, all irreducible representation will be one dimensional. Such an irreducible representation will be given by a differentiable map

$$\pi: U(1) \rightarrow GL(1, \mathbb{C}) = \{z \neq 0\}$$

A differentiable map π that is a representation of $U(1)$ must satisfy homomorphism and periodicity properties which can be used to show:

Theorem 2.3 All irreducible representations of the group $U(1)$ are unitary, and given by

$$\pi_k: e^{i\theta} \in U(1) \rightarrow \pi_k(\theta) = e^{ik\theta} \in U(1) \subset GL(1, \mathbb{C}) \cong \mathbb{C}^*$$

for $k \in \mathbb{Z}$.

$$\pi(1) = e \quad \left(\begin{array}{l} e = ee \\ \pi(e) = \pi(e)\pi(e) \\ \pi(e) = e \end{array} \right)$$

proof: We will write the π_k as a function of an angle $\theta \in \mathbb{R}$.
 So satisfying the periodicity property $\pi_k(2\pi) = \pi_k(0) = 1$
 $e^{i0} = e^{i2\pi k} \quad e^{i0} = 1 \Rightarrow 2\pi k = 0$

Since it is a representation, π will satisfy the homomorphism property

$$\pi_k(\theta_1 + \theta_2) = \pi_k(\theta_1) \pi_k(\theta_2)$$

$$\begin{aligned} \pi'_k(\theta) &= \lim_{\Delta\theta \rightarrow 0} \frac{\pi_k(\theta + \Delta\theta) - \pi_k(\theta)}{\Delta\theta} = \pi_k(\theta) \lim_{\Delta\theta \rightarrow 0} \frac{\pi_k(0) - \pi_k(\theta)}{\Delta\theta} \\ &= \pi_k(\theta) \pi'_k(0) \end{aligned}$$

$$\Rightarrow \pi'_k(\theta) = e^{\pi'_k(0)\theta}$$

$$\pi_k(2\pi) = e^{\pi'_k(0)2\pi} = \pi_k(0) = 1$$

$$\Rightarrow \pi'_k = ik. \text{ for } k \in \mathbb{Z}. \text{ and } f = \pi_k \text{ for some integer } k.$$

Remark: The representations we have found are all unitary, with π_k taking values in $U(1) \subset \mathbb{C}^*$. The complex numbers $e^{ik\theta}$ satisfy the condition to be a unitary 1×1 matrix, since

$$(e^{ik\theta})^{-1} = e^{-ik\theta} = \overline{e^{ik\theta}}$$

If the representation of $U(1)$ on H is irreducible, by theorem 2.2 it must be one dimensional with $H = \mathbb{C}$. By theorem 2.3, it must be of the form $(\lambda q, \mathbb{C})$ for some $q \in \mathbb{Z}$. In this case the self adjoint operator Q is multiplication of elements of H by the integer q . (10)

For a general $U(1)$ representation, by theorems 2.1 and 2.3 we have
 Unitary $\Rightarrow$ direct sum $\Rightarrow$ irreducible $\Rightarrow$ one dimension.
 $H = H_{q_1} \oplus H_{q_2} \oplus \dots \oplus H_{q_n}$

for some set of integers $q_1, q_2, \dots, q_n$. (n is the dimension of H , the q_i may not be distinct).

With the above preparations, we can define first example of a quantum mechanical observable Q . Q is a self-adjoint operator named charge operator on H ,

Definition. The charge operator Q for the $U(1)$ representation (λ, H) is the self-adjoint linear operator on H that acts by multiplication by q_i on the irreducible sub-representation H_{q_i} . Taking basis elements in H_{q_i} it acts on H as the matrix

$$Q = \begin{pmatrix} q_1 & 0 & \dots & 0 \\ 0 & q_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & q_n \end{pmatrix}$$

$\mathbb{Q} | \psi \rangle = 2 | \psi \rangle$ if $H = H_1 \oplus H_0 \oplus H_2$ and $\psi \in H_2$.

If $| \psi \rangle = a | \psi_1 \rangle + b | \psi_2 \rangle$ & $| \psi_1 \rangle = q_1 | \psi_1 \rangle$ & $| \psi_2 \rangle = q_2 | \psi_2 \rangle$.

Then

$\mathbb{Q} | \psi \rangle = a q_1 | \psi_1 \rangle + b q_2 | \psi_2 \rangle$ (no total change, but have Born rule)

$$\pi(e^{i\theta}) = e^{i\mathbb{Q}\theta} = \begin{pmatrix} e^{iq_1\theta} & 0 & \dots & 0 \\ 0 & e^{iq_2\theta} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{iq_n\theta} \end{pmatrix} \text{ CUM/CGL(N)}$$

differential π' : $i\theta \in i\mathbb{R} \rightarrow \pi'(i\theta) = i\mathbb{Q}\theta$.

lie group $U(1)$
lie algebra $i\mathbb{R}$

lie group $\pi: U(1) \rightarrow U(n)$
 $e^{i\theta} \rightarrow e^{i\mathbb{Q}\theta}$
lie algebra $\pi': i\theta \rightarrow i\mathbb{Q}\theta$

"The tangent space at the identity of a Lie group is called a "Lie algebra", "

$\mathbb{Q} | \psi \rangle = q_j | \psi \rangle$ if $| \psi \rangle \in H_{q_j}$ at $t=0$. What is dynamics of observable operators?

Before, we know Schrödinger equation Planck constant.

$$\frac{d}{dt} | \psi(t) \rangle = -i H | \psi(t) \rangle \quad (\hbar=1)$$

$$| \psi(t) \rangle = U(t) | \psi(0) \rangle$$

where $U(t) = e^{-itH} = 1 - itH + \frac{(it)^2}{2!} H^2 t^2 - \dots$

If $[Q, H] = 0$. $[O_1, O_2] = O_1 O_2 - O_2 O_1$

the $[H^K, Q] = H^K Q - Q H^K = 0 \Rightarrow [U(t), Q] = 0$

$\Rightarrow Q |\psi(t)\rangle = Q U(t) |\psi(0)\rangle = U(t) Q |\psi(0)\rangle = U(t) q_j |\psi(0)\rangle$

Remark: If an observable commutes with the $= q_j |\psi(t)\rangle$

Hamiltonian observable, one gets a conservation law.

This conservation law says that if one starts in a state with a well-defined numerical value for the observable (an eigenvector for the observable operator), one will remain in such a state with the value not changing.

In general $[Q, H] \neq 0$.

Take Home message.

- Irreducible representations π are one dimensional and characterized by their derivative π' at the identity.
- An arbitrary representation π of $U(1)$ is of the form

$$\pi(e^{i\theta}) = e^{i\theta Q};$$

where Q is a matrix with eigenvalues a set of integers.

Self-adjoint observable.

- If $[Q, H] = 0$, the $U(1)$ group acts on the state space as "symmetries". In this case the q_j will be "conserved quantities", do not change as the states evolve in time.