Chapter 9. Tensor products, Entanglement, and addition of Spin

- Multi particle system

① Tensor product ② Entanglement ③ Tensor product of representation ④ Multi linear form

9.1 Tensor products

V, W be two vector spaces with bases B_V, B_W.

$V \otimes W$ is a vector space with a basis $\{ v_i \otimes w_j \mid v_i \in B_V, w_j \in B_W \}$.

For $v = \sum_i c_i v_i$, $w = \sum_j d_j w_j$, define $v \otimes w = \sum_{i,j} c_i d_j (v_i \otimes w_j)$.

※ It is independent of choice of bases up to isomorphism.

properties (Bilinearity)

1. ~~Distributive laws~~ $(v \otimes (c_1 w_1 + c_2 w_2) = c_1 v \otimes w_1 + c_2 v \otimes w_2, \cdots)$
2. $V \otimes W \simeq W \otimes V$, $V \otimes (W \otimes U) \simeq (V \otimes W) \otimes U$
3. For $A \in L(V)$, $B \in L(W)$, define $A \otimes B \in L(V \otimes W)$ by $(A \otimes B)(v \otimes w) = Av \otimes Bw$.
4. $\dim V \otimes W = \dim V \dim W$. cf) $\dim V \oplus W = \dim V + \dim W$.
5. $V^* \otimes W \cong L(V, W)$, $l \otimes w \mapsto (v \in V \mapsto l(v) w \in W)$.

9.2 Composite quantum systems and tensor product

(H_i, O_i) be quantum state space and observable set.

$H_1 \otimes H_2$ describes composite quantum system. For non-interacting system, operators $A \otimes Id_2 + Id_1 \otimes B$ are used.

System of n-identical particles : $H^{\otimes n} = \underbrace{H \otimes H \cdots \otimes H}_{n}$. For symmetric group S_n, it has a representation on $H^{\otimes n}$ by permutation :

$\pi(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$. In general it is reducible.

A fundamental axiom of quantum mechanics implies that for identical particles, physical states' ~~should~~ ray $\{ c\psi : c \in \mathbb{C} \}$ should be S_n-invariant.

In other words, $\pi(\sigma)\psi = c(\sigma)\psi$, $c(\sigma_1)c(\sigma_2) = c(\sigma_1\sigma_2)$.

Thus $c: S_n \to \mathbb{C}^\times$ is a one dimensional representation of S_n.

There are two such representations. So for $v \in \mathcal{H}^{\otimes n}$, it is called

① Bosonic if $\pi(\sigma)v = v$. Such (invariant) set is denoted by $S^n(\mathcal{H})$.

② Fermionic if $\pi(\sigma)v = (-1)^{|\sigma|}v$. Such (invariant) set is denoted by $\wedge^n(\mathcal{H})$.

※ For Fermions, we can easily show Pauli principle:

$\pi((12))(w\otimes w) = -w\otimes w = w\otimes w = 0 \quad \to\leftarrow$

9.3 Entanglement

Note that $\{fg : f: X\to\mathbb{R},\ g: Y\to\mathbb{R}\} \subsetneq \{F: X\times Y\to\mathbb{R}\}$. Likewise

$\{v\otimes w : v\in V, w\in W\} \subsetneq V\otimes W$.

Def Decomposable / Entanglement

$x \in V\otimes W$ is called decomposable if $x = v\otimes w$ for some $v\in V$, $w\in W$.

If not, it is called indecomposable.

For $\mathcal{H}_1\otimes\mathcal{H}_2$, indecomposable state is also called an entangled state.

9.4 Tensor products of representations

Recall

Group representation

$\pi: G \to GL(V)$,

$\pi(g_1g_2) = \pi(g_1)\pi(g_2)$

Lie algebra representation

$\pi': \mathfrak{g} \to \mathfrak{gl}(V)$ $(\mathfrak{g} = T_eG,\ \mathfrak{gl}(V) = End(V) = L(V))$

$\pi'(X) = \frac{d}{dt}\pi(e^{tX})\Big|_{t=0}$.

Def Tensor product representation.

(π_V, V), (π_W, W) : representations of G. $(\pi_{V\otimes W}, V\otimes W)$ is defined by

$\pi_{V\otimes W}(g)(v\otimes w) = \pi_V(g)v \otimes \pi_W(g)w$.

$\pi_V: G \to GL(V)$

$\pi_W: G \to GL(W)$

$\pi_{V\otimes W}: G \to GL(V\otimes W)$

If G is a Lie group, corresponding Lie algebra representation of $\pi_{V\otimes W}$ is

$$\pi'_{V\otimes W}(X)(V\otimes W) = \frac{d}{dt}\Big|_{t=0} \pi_{V\otimes W}(e^{tX})(V\otimes W) = \frac{d}{dt}\Big|_{t=0} \pi_V(e^{tX})(v)\otimes \pi_W(e^{tX})(w)$$

by bilinearity $\rightarrow$

$$= \left(\frac{d}{dt}\Big|_{t=0} \pi_V(e^{tX})v\right)\otimes \pi_W(e^{tX})w + \pi_V(e^{tX})v \otimes \left(\frac{d}{dt}\Big|_{t=0} \pi_W(e^{tX})w\right)$$

$$= \pi'_V(X)v \otimes w + v \otimes \pi'_W(X)w$$

$$= (\pi'_V \otimes Id_W + Id_V \otimes \pi'_W)(v\otimes w).$$

If we know irreducible decomposition of (π_V, V) and (π_W, W), can we get it for $(\pi_{V\otimes W}, V\otimes W)$, too? It is enough when (π_V, V), (π_W, W) are irreducible. For $G = SU(2)$, it can be answered.

Thm Clebsch-Gordan decomposition.
For irr. representations $(\pi_{V^{n_1}}, V^{n_1})$, $(\pi_{V^{n_2}}, V^{n_2})$,

$$(\pi_{V^{n_1}\otimes V^{n_2}}, V^{n_1}\otimes V^{n_2}) \cong (\pi_{n_1+n_2}, V^{n_1+n_2}) \oplus (\pi_{n_1+n_2-2}, V^{n_1+n_2-2}) \oplus \cdots \oplus (\pi_{|n_1-n_2|}, V^{|n_1-n_2|}).$$

One way to prove this is to use raising/lowering / Casimir operator of previous section, but there is an easier method using 'character'.

Def Character.
The character of (π, V) is $\chi_V : G \to \mathbb{C}$ defined by

$$\chi_V(g) = Tr(\pi(g)).$$

Note that trace is invariant under conjugate. One can check that

$$\chi_{V\oplus W} = \chi_V + \chi_W, \quad \chi_{V\otimes W} = \chi_V \chi_W.$$

(proof of 2nd equation) Take bases $\{v_1, \cdots v_n\}$, $\{w_1, \cdots w_m\}$, $\{v_i \otimes w_j\}_{i,j}$.

$$\pi_{V\otimes W}(g)(v_i\otimes w_j) = \sum_{k,l} a_{ki} b_{lj} v_k \otimes w_l$$

$$\pi_V(g)v_i = \sum_k a_{ki} v_k$$

$$\pi_W(g)w_j = \sum_l b_{lj} w_l$$

For trace we need $a_{ii} b_{jj}$.

$$\Rightarrow \chi_{V\otimes W}(g) = Tr(\pi_{V\otimes W}(g)) = \sum_{i,j} a_{ii} b_{jj} = \sum_i a_{ii} \sum_j b_{jj} = Tr(\pi_V(g))\, Tr(\pi_W(g)) = \chi_V(g)\chi_W(g)$$

The key to show the theorem is that for each irr. representation the character is distinguishable. Note that any element of $SU(2)$ can be conjugated to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. Since χ is conjugate invariant, it is enough to compute for this case.

<u>Thm</u> Weyl character formula

$$\chi_{V^n}\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) = e^{in\theta} + e^{i(n-2)\theta} + \cdots + e^{-in\theta} = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

pf) Recall that the basis of V^n is $\{z_1^{n-j} z_2^j\}_{j=0}^{n}$ (homogeneous polynomial).

For the given matrix, $z_1^{n-j} z_2^j \mapsto e^{i(n-2j)\theta} z_1^{n-j} z_2^j$. Thus each basis is eigenvector and eigenvalues are $e^{i(n-2j)\theta}$, $j = 0 \sim n$.

Trace is the sum of eigenvalues, so it proves the claim.

Now we can show the Clebsch-Gordan decomposition theorem.

pf) $\chi_{V^{n_1} \otimes V^{n_2}} = \chi_{V^{n_1}} \chi_{V^{n_2}}$ $\quad \leftarrow g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$

$$= \left(e^{in_1\theta} + e^{i(n_1-2)\theta} + \cdots + e^{-i(n_1)\theta}\right)\left(\frac{e^{i(n_2+1)\theta} - e^{-i(n_2+1)\theta}}{e^{i\theta} - e^{-i\theta}}\right)$$

Assume $n_1 < n_2$

$$= \frac{1}{e^{i\theta} - e^{-i\theta}}\left(\left[e^{i(n_1+n_2+1)\theta} - e^{-i(n_1+n_2+1)\theta}\right] + \cdots + \left[e^{i(n_2-n_1)\theta} - e^{-i(n_2-n_1)\theta}\right]\right)$$

$$= \chi_{V^{n_1+n_2+1}} + \cdots + \chi_{V^{|n_1-n_2|}} = \chi_{V^{n_1+n_2-1} \oplus \cdots \oplus V^{|n_1-n_2|}}.$$

<u>Example</u>

$V^1 \otimes V^1 \cong V^2 \oplus V^0$. Let's use $|\uparrow\rangle, |\downarrow\rangle$ for the basis of V^1 to see what exactly corresponds to V^0 part, V^2 part.

By computation, $\frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)$ is invariant under $SU(2)$.

Its span is one dimensional, so it is an irr. component corresponding to V^0.

Note that it is a Fermion. Other component V^2 has $\{|\uparrow\rangle \otimes |\uparrow\rangle, |\downarrow\rangle \otimes |\downarrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle)\}$ as a basis, and is Bosonic.

9.5 Bilinear forms and tensor products

Def Bilinear forms.

$B : V \times V \to K$ ($\mathbb{R}$ or $\mathbb{C}$) s.t. each entry is linear.

If $B(v_1, v_2) = B(v_2, v_1)$ it is called symmetric, and if $B(v_1, v_2) = -B(v_2, v_1)$ it is called antisymmetric.

Thm Space of bilinear form is isomorphic to $V^* \otimes V^*$.

pf) Map by $\alpha_1 \otimes \alpha_2 \in V^* \otimes V^* \mapsto B$ ($B(v, v') = \alpha_1(v)\alpha_2(v')$).

Then extend this to whole $V^* \otimes V^*$.

Note that bilinear form can also be written in terms of quadratic function:

$B(v, v') = v \cdot Bv'$, $B_{ij} = B(v_i, v_j)$.

9.6 Symmetric and Antisymmetric multilinear forms

Recall $S^n(V)$ is a subset of $V^{\otimes n}$ s.t. $\pi(\sigma)v = v$ $\forall \sigma \in S_n$.

$\Lambda^n(V)$ is a subset of $V^{\otimes n}$ s.t. $\pi(\sigma)v = (-1)^{|\sigma|} v$ $\forall \sigma \in S_n$.

Define $S^*(V^*) = \oplus_n S^n(V^*)$, $\Lambda^*(V^*) = \oplus_n \Lambda^n(V^*)$, and call them as the space of symmetric/antisymmetric multilinear forms.

Note that the elements of basis is n-linear ($(\alpha_1 \otimes \cdots \otimes \alpha_n)(v_1 \otimes \cdots v_n)$
$= \alpha_1(v_1) \cdots \alpha_n(v_n)$)

We can define product in each spaces:

S^* : $(\alpha_1 \otimes \cdots \alpha_k)(\alpha_{k+1} \otimes \cdots \alpha_n) := \frac{1}{n!} \sum_{\sigma \in S_n} \alpha_{\sigma(1)} \otimes \cdots \alpha_{\sigma(n)}$ (corresponds to multiplication of polynomials)

Λ^* : $(\alpha_1 \otimes \cdots \alpha_k) \wedge (\alpha_{k+1} \otimes \cdots \alpha_n) := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{|\sigma|} \alpha_{\sigma(1)} \otimes \cdots \alpha_{\sigma(n)}$.

Also we have $(S^*(V))^* \cong S^*(V^*)$, $(\Lambda^*(V))^* \simeq \Lambda^*(V^*)$

Chapter 10. Momentum and the free particle

De Broglie: Matter with momentum p can behave like a wave

$e^{ik\cdot q}$, where $p=\hbar k$, q is a position.

Applying $-i\hbar\nabla$ to above function gives eigenvalue $\hbar k = p$.

$\therefore$ This is called the momentum operator.

① Representation of $(\mathbb{R},+)$ ② Translation in time/space

③ Schrödinger equation for free particle

10.1 The group $\mathbb{R}$ and its representations

$(\mathbb{R},+)$ is a Lie group.

Thm Irreducible representation of $(\mathbb{R},+)$ are labeled by $c\in\mathbb{C}$:

$\pi_c(a) = e^{ca}$. It is unitary when $c\in i\mathbb{R}$.

proof is almost the same with $U(1)$ which was done in thm 2.3.

Next, think of representation of $(\mathbb{R},+)$ on $V = L^2(\mathbb{R})$ or $\{f:\mathbb{R}\to\mathbb{C}\}$, by $(\pi(a)f)(x) = f(x-a)$. The corresponding Lie algebra representation is

$$\pi'(a)f = \frac{d}{dt}\Big|_{t=0} \pi(ta)f = \frac{d}{dt}\Big|_{t=0} f(\cdot - ta) = -a\frac{d}{dq}f$$

$$\Rightarrow \pi'(a) = -a\frac{d}{dq}, \quad \pi'(1) = -\frac{d}{dq}.$$

$\therefore$ Note that for 1-d, exponential map is $\overset{a}{\downarrow}\ ta$. This can be seen by

$$(\mathbb{R},+) \leftrightarrow \left\{\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}; \cdot\right\} \Rightarrow e^{t\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}} = \begin{pmatrix}1 & ta\\ 0 & 1\end{pmatrix}.$$

10.2 Translations in time and space

Assume $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. Translation in time relates with Hamiltonian.

↑ time only dependent state

$$\begin{cases} \text{Time-independent reversible dynamics} \\ \text{probability preserving} \end{cases} \Leftrightarrow \begin{cases} i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \\ H \text{ is time-independent} \end{cases}$$

($\Rightarrow$) Time independence + reversible gives $U(t+s) = U(t)U(s)$ $\forall t, s \in \mathbb{R}$.

Thus U is a representation of $(\mathbb{R}, +)$. Then by thm, $U(t) = e^{CA}$.

Probability preserving implies $|U(t)| = 1$, i.e. unitary.

$\therefore U(t) = e^{-itH/\hbar}$.

($\Leftarrow$) Integration gives $|\psi(t)\rangle = e^{-itH/\hbar}|\psi(0)\rangle =: U(t)|\psi(0)\rangle$.

Then by computation $|U(t)| = 1$, $U(t+s) = U(t)U(s)$.

Note that $-iH$ is in Lie algebra, so that H is self-adjoint. ← observable

Def Momentum operator

$\mathcal{H} = L^2(\mathbb{R}^3)$, denote state variables as q_i, $i = 1 \sim 3$.

Momentum operators are defined by $P_i = -i\hbar \frac{\partial}{\partial q_i}$, $i = 1 \sim 3$.

Here $-i$ is for self-adjoint, $\hbar$ is for scaling and unit.

Note that it is a Lie algebra representation of spatial translation.

10.3 The energy-momentum relation and the Schrödinger equation for a free particle

By special relativity, $E^2 = |p|^2c^2 + m^2c^4$. In non-relativistic limit, we may approximate by $E \simeq \frac{|p|^2}{2m} + mc^2$ and drop the constant mc^2.

Since eigenvalue of Hamiltonian operator is interpreted as the energy,

let $H = \frac{|\mathbf{P}|^2}{2m} = -\frac{\hbar^2}{2m}\left(\partial_{q_1}^2 + \partial_{q_2}^2 + \partial_{q_3}^2\right)$.

Schrödinger equation becomes

$i\hbar \frac{\partial}{\partial t}\psi(q,t) = -\frac{\hbar^2}{2m}\nabla^2\psi(q,t)$. Separation of variables gives the

basis solutions: $\underbrace{e^{ik\cdot q}\, e^{-\frac{i}{\hbar}tE}}_{\psi(q,t)=}$, $E_{|k\rangle} = \frac{\hbar^2|k|^2}{2m}$.

$|k\rangle := e^{ik\cdot q}$ is an eigenstate of both P_j and H:

$P_j|k\rangle = \hbar k_j|k\rangle$, $H|k\rangle = \frac{\hbar^2}{2m}|k|^2|k\rangle = E_{|k\rangle}|k\rangle$.

$= P_j|k\rangle$

Note that $|k\rangle \notin \mathcal{H} = L^2(\mathbb{R}^3)$. So actual states are represented as

~~$\psi = \int c(k)\, e^{ik\cdot q}\, dk$~~

$\psi(q,t) = \int c(k)\, e^{ik\cdot q}\, e^{-it\hbar\frac{|k|^2}{2m}}\, dk$, with appropriate coefficient $c(k)$.