

① Chapter 3. Two-state Systems and $SU(2)$.

Qubits: $\mathbb{C}^2 \rightarrow$ Pauli matrices ($\sigma_1, \sigma_2, \sigma_3$) $\rightarrow$ observables

$\rightarrow SU(2) \rightarrow$ Time evolution.

Qubits : $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \underset{\substack{\downarrow \\ \text{ket}}}{|0\rangle} + \beta \underset{\substack{\downarrow \\ \text{Basis}}}{|1\rangle} \in \mathbb{C}^2, \quad |\alpha|^2 + |\beta|^2 = 1.$

Pauli matrices : $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

Observables : $A = c_0 I + \mathbf{c} \cdot \boldsymbol{\sigma}$ (self-adjoint matrices)
 $M = M^\dagger$

Unitary transformation $SU(2) = \left\{ U = e^{i\theta \mathbf{n} \cdot \boldsymbol{\sigma}} : \theta \in \mathbb{R}, \mathbf{n} \in S^2 \right\}.$

Schrödinger equation : $|\psi(t)\rangle = e^{-itH} |\psi(0)\rangle, \quad H = h_0 I + \mathbf{h} \cdot \boldsymbol{\sigma}.$

Q : How do the observables of the simplest nontrivial quantum system determine its continuous transformations and its time evolution?

"2-d state space."

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

— α : ~~prob~~ probability amplitudes. not measurement outcomes.

— $|\alpha|^2$: probability of $|0\rangle$.

② eg: $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \Rightarrow \langle + | - \rangle = 0$.

$\swarrow$
 Basis

"An observable supplies the eigenbasis and eigenvalues that determine the measurement question".

"Pauli matrices: the observables of a qubit".

$\downarrow$
self-adjoint matrix.

$$M = M^\dagger \Rightarrow M = \begin{pmatrix} a & z \\ \bar{z} & d \end{pmatrix} \quad a, d \in \mathbb{R} \quad z \in \mathbb{C} \quad \begin{matrix} \nearrow \\ \text{Re } z \quad \text{Im } z \end{matrix}$$

Basis: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\Rightarrow \sigma_j^2 = I \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1 \quad \leftarrow \text{outcomes}$$

Observable	Eigenvector for +1	Eigenvector for -1
σ_1	$(0\rangle + 1\rangle)/\sqrt{2}$	$(0\rangle - 1\rangle)/\sqrt{2}$
σ_2	$(0\rangle + i 1\rangle)/\sqrt{2}$	$(0\rangle - i 1\rangle)/\sqrt{2}$
σ_3	$ 0\rangle$	$ 1\rangle$

example: "electron spin"

prepare electron $\rightarrow$ send it into a Stern-Gerlach magnet
 $\rightarrow$ two detectors.

eg: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ σ_3 outcome +1 with probability $|\alpha|^2$
 outcome -1 with $|\beta|^2$

expectation: $\langle \psi | \sigma_3 | \psi \rangle = |\alpha|^2 - |\beta|^2$

σ_1 : $\left| \frac{\alpha + \beta}{\sqrt{2}} \right|^2$ "+1" $\left| \frac{\alpha - \beta}{\sqrt{2}} \right|^2$ "-1"

③ "no common Pauli eigenvector." no 0 eigenvalue.

$$[\sigma_j, \sigma_k] = \sigma_j \sigma_k - \sigma_k \sigma_j = \pm 2i \sigma_l \quad \text{--- invertible.}$$

$$\begin{aligned} [\sigma_1, \sigma_2] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i \sigma_3. \end{aligned}$$

if $\exists v \in \mathbb{C}^2$ s.t. $\sigma_j v = \lambda_1 v$, $\sigma_k v = \lambda_2 v$ ($\lambda_1 \neq 0, \lambda_2 \neq 0$)

$$\begin{aligned} \Rightarrow \lambda_2 \sigma_j v &= \lambda_1 \sigma_k v \Rightarrow \lambda_2 \sigma_k \sigma_j v = \lambda_1 \sigma_k^2 v = \lambda_1 \lambda_2 \sigma_k v \\ &= \lambda_1 \lambda_2 v \end{aligned}$$

$$\Rightarrow \sigma_k \sigma_j v = \frac{\lambda_1}{\lambda_2} v \quad \sigma_j \sigma_k v = \frac{\lambda_2}{\lambda_1} v$$

$$\Rightarrow (\sigma_k \sigma_j - \sigma_j \sigma_k) v = \frac{\lambda_1}{\lambda_2} v - \frac{\lambda_2}{\lambda_1} v = 0$$

$$\Rightarrow v = 0.$$

"Exponentials: from observables to unitary transformations."

??
"An observable describes a measurement, but it does not by itself describe a continuous change on the state."

$A = A^\dagger \Rightarrow U(\theta) = e^{i\theta A}$ ($\forall \theta \in \mathbb{R}$) is unitary $\Rightarrow$ probability is preserved.
 $UU^\dagger = U^\dagger U = I$

$$\begin{aligned} (e^{i\theta A})^\dagger &= (I + i\theta A + \frac{(i\theta A)^2}{2!} + \dots)^\dagger \\ &= (I - i\theta A^\dagger + \frac{(i\theta A^\dagger)^2}{2!} + \dots)^\dagger \\ &= I - i\theta A + \frac{(-i\theta A)^2}{2!} + \dots = e^{-i\theta A} = (e^{i\theta A})^{-1} \end{aligned}$$

unitary.

Lemma: $e^{i\theta \sigma_j} = \cos \theta I + i \sin \theta \sigma_j$

Proof: Since $\sigma_j^{2n} = I$, $\sigma_j^{2n+1} = \sigma_j$, $(i)^{2n} = (-1)^n$, $(i)^{2n+1} = (-1)^n i$.

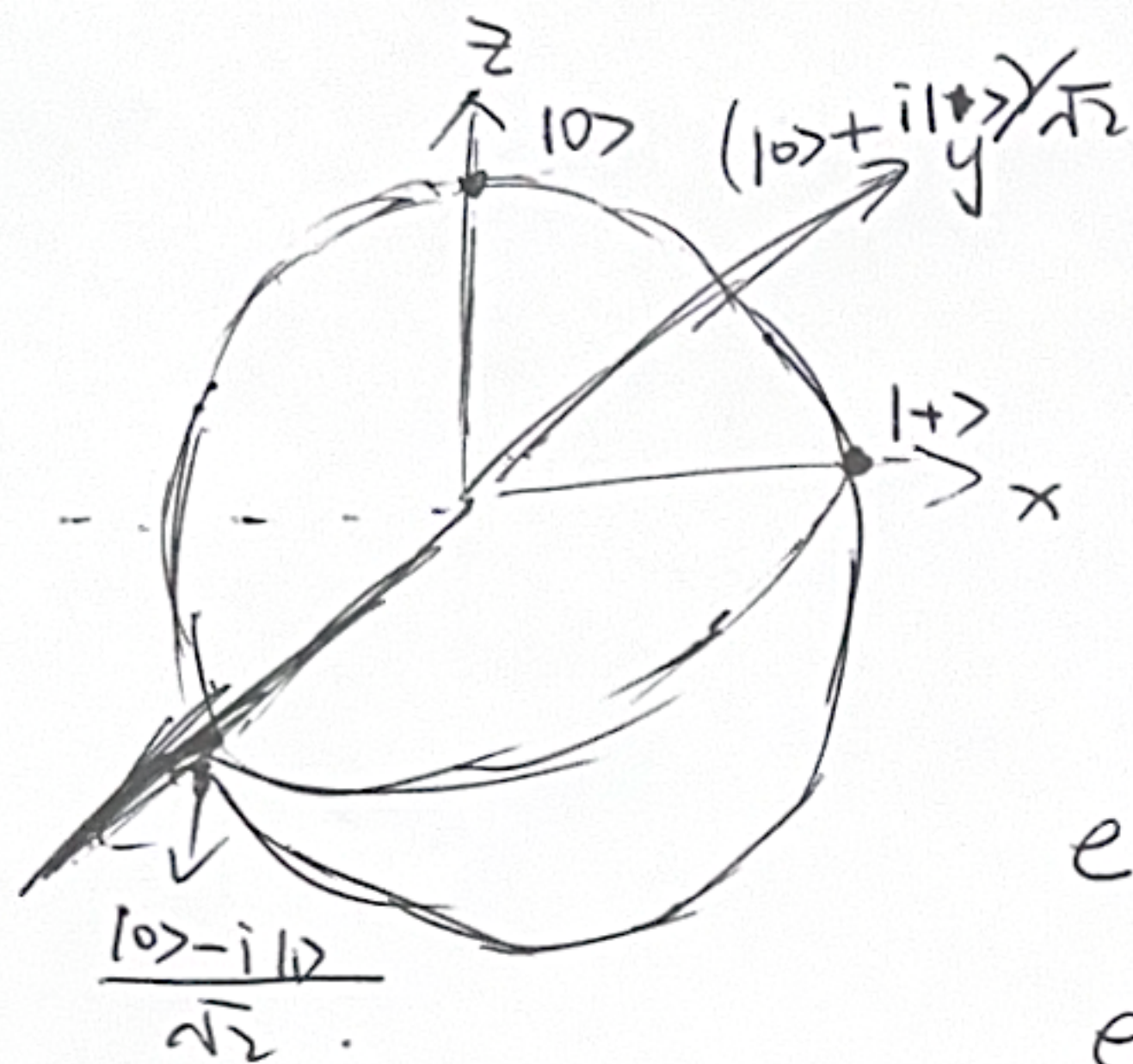
$$\begin{aligned} \Rightarrow e^{i\theta \sigma_j} &= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} I + \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} \sigma_j \\ &= \cos \theta I + i \sin \theta \sigma_j \end{aligned}$$

$$(k) \quad e^{i\theta\sigma_1} = \cos\theta I + i\sin\theta \sigma_1 = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & i\sin\theta \\ i\sin\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$$

$$e^{i\theta\sigma_2} = \cos\theta I + i\sin\theta \sigma_2 = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$e^{i\theta\sigma_3} = \cos\theta I + i\sin\theta \sigma_3 = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} i\sin\theta & 0 \\ 0 & -i\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

x -axis $\leftrightarrow \sigma_1$ y -axis $\leftrightarrow \sigma_2$ z -axis $\leftrightarrow \sigma_3$



$$e^{i\theta\sigma_j} e^{i\phi\sigma_j} = e^{i(\theta+\phi)\sigma_j} \Rightarrow e^{i(\theta+2\pi)\sigma_j} = e^{i\theta\sigma_j}$$

$\{ e^{i\theta\sigma_j} : \theta \in \mathbb{R} \}$ isomorphic to $U(1)$

$$e^{i\theta I} = e^{i\theta} I \quad e^{i\phi} I |0\rangle = e^{i\phi} |0\rangle$$

$$e^{i\theta\sigma_3} |0\rangle = e^{i\theta} |0\rangle \quad e^{i\theta\sigma_3} |1\rangle = e^{-i\theta} |1\rangle$$

"anticommutate" : $[\cdot, \cdot]_+$

$$[A, B]_+ = AB + BA$$

$$\Rightarrow [\sigma_j, \sigma_k]_+ = \sigma_j \sigma_k + \sigma_k \sigma_j = \begin{cases} 0 & k \neq j \\ 2I & k = j \end{cases} \triangleq 2\delta_{jk} I$$

For $v = (v_1, v_2, v_3) \in \mathbb{R}^3$, set

$$v \cdot \sigma = v_1 \sigma_1 + v_2 \sigma_2 + v_3 \sigma_3 = \begin{pmatrix} 0 & v_1 \\ v_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iv_2 \\ iv_2 & 0 \end{pmatrix} + \begin{pmatrix} v_3 & 0 \\ 0 & -v_3 \end{pmatrix} = \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix}$$

$$\Rightarrow (v \cdot \sigma)^2 = (v_1 \sigma_1 + v_2 \sigma_2 + v_3 \sigma_3)(v_1 \sigma_1 + v_2 \sigma_2 + v_3 \sigma_3)$$

$$= \sum_j v_j^2 I + \sum_{j < k} v_j v_k \underbrace{[\sigma_j, \sigma_k]_+}_{=0}$$

$$= |v|^2 I$$

$\Rightarrow$ If $n \in S^2$ (i.e. $|n|=1$) $\Rightarrow n \cdot \sigma$ satisfying $(n \cdot \sigma)^2 = I$

$$\begin{aligned}
 (5) \Rightarrow e^{i\theta n \cdot \sigma} &= I + i\theta n \cdot \sigma + \frac{(i\theta n \cdot \sigma)^2}{2!} + \dots \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} I + \sum_{n=0}^{\infty} \frac{i(-1)^n \theta^{2n+1}}{(2n+1)!} n \cdot \sigma \\
 &= \cos \theta I + i \sin \theta n \cdot \sigma
 \end{aligned}
 \left| \begin{array}{l} (i\theta n \cdot \sigma)^{2n} = (-1)^n \theta^{2n} \\ (i\theta n \cdot \sigma)^{2n+1} = (-1)^n i \theta^{2n+1} n \cdot \sigma \end{array} \right.$$

Claim : $SU(2) = \{ e^{i\theta n \cdot \sigma} : \forall n \in S^2 \}$
 $= \{ U \in M_2(\mathbb{C}) : U^\dagger U = I, \det U = 1 \}$.

$\Rightarrow$ prove $SU(2) \supseteq \{ \quad \}$.

$$\begin{aligned}
 1^\circ (e^{i\theta n \cdot \sigma})^\dagger \cdot (e^{i\theta n \cdot \sigma}) &= (\cos \theta I + i \sin \theta n \cdot \sigma)^\dagger (\cos \theta I + i \sin \theta n \cdot \sigma) \\
 &= (\cos \theta I - i \sin \theta n \cdot \sigma) (\cos \theta I + i \sin \theta n \cdot \sigma) \\
 &= \cos^2 \theta I + \sin^2 \theta (n \cdot \sigma)^2 \\
 &= (\cos^2 \theta + \sin^2 \theta) I = I
 \end{aligned}$$

$$\begin{aligned}
 2^\circ \det(e^{i\theta n \cdot \sigma}) &= \det(\cos \theta I + i \sin \theta n \cdot \sigma) \\
 &= \det \left(\begin{pmatrix} \cos \theta & 0 \\ 0 & \cos \theta \end{pmatrix} + \begin{pmatrix} i \sin \theta n_3 & i \sin \theta n_1 + i \sin \theta n_2 \\ i \sin \theta n_1 - i \sin \theta n_2 & -i \sin \theta n_3 \end{pmatrix} \right) \\
 &= \begin{vmatrix} \cos \theta + i \sin \theta n_3 & i \sin \theta n_1 + i \sin \theta n_2 \\ i \sin \theta n_1 - i \sin \theta n_2 & \cos \theta - i \sin \theta n_3 \end{vmatrix} \\
 &= \cos^2 \theta + \sin^2 \theta (n_1^2 + n_2^2 + n_3^2) = 1
 \end{aligned}$$

$\Leftarrow$ prove $SU(2) \subseteq \{ \quad \}$.

$1^\circ U \in SU(2) \Leftrightarrow U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, a, b \in \mathbb{C}$.

(i) $U^\dagger U = \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & \bar{a}b - b\bar{a} \\ \bar{b}a - a\bar{b} & |b|^2 + |a|^2 \end{pmatrix} = I$ (since $|a|^2 + |b|^2 = 1$).

$\det(U) = \begin{vmatrix} a & b \\ -\bar{b} & \bar{a} \end{vmatrix} = |a|^2 + |b|^2 = 1$.

(ii) set $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow U^\dagger = \begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix} = U^\dagger \Rightarrow \begin{cases} \bar{a} = d \\ b = -\bar{c} \end{cases}$
 with $ad - bc = 1$

⑥ For $U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$. $a, b \in \mathbb{C}$. $|a|^2 + |b|^2 = 1$

~~(i) $b=0 \Rightarrow U = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$~~

Set $a = a_0 + ia_3$. $b = a_2 + ia_1$, $a = (a_0, a_1, a_2, a_3)$,

$\Rightarrow U = \begin{pmatrix} a_0 + ia_3 & a_2 + ia_1 \\ -a_2 + ia_1 & a_0 - ia_3 \end{pmatrix} = a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i a_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i a_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + i a_3 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$\boxed{SU(2) \cong S^3} = a_0 I + i (a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3)$

$= a_0 I + i a \cdot \sigma$, with $a_0^2 + |a|^2 = 1$.

(i) if $a=0$. $\Rightarrow U = I = e^{i \cos(\theta) n \cdot \sigma}$

(ii) if $a \neq 0$, $\Rightarrow \cos \theta = a_0$, $\sin \theta = |a|$, $n = \frac{a}{|a|}$, $n = 0$,

$\Rightarrow U = \cos \theta I + i \sin \theta n \cdot \sigma = e^{i \theta n \cdot \sigma}$

"Adding the scalar phase gives $U(2)$."

$U(2) = \{ W : W^\dagger W = I \}$. $\Rightarrow |\det W| = 1$.

choose $\phi \in \mathbb{R}$. s.t. $e^{2i\phi} = \det W$.

Set $V = e^{-i\phi} W$. $\Rightarrow V^\dagger V = W^\dagger W = I$.

$\det V = e^{-2i\phi} \det W = 1$

$\Rightarrow V \in SU(2)$. $\Rightarrow V = e^{i \theta n \cdot \sigma}$.

$\Rightarrow W = \exp (i \phi I + i \theta n \cdot \sigma)$

"Commutation relations and noncommutative geometry"

$[\sigma_j, \sigma_k] = 2i \sum_l \epsilon_{jkl} \sigma_l$.

repeated $\Rightarrow$ zero

changing two indices . reverses the sign .

⑦ Since $AB = \frac{1}{2}[A, B]_+ + \frac{1}{2}[A, B]$.

$$\Rightarrow \sigma_j \sigma_k = \delta_{jk} I + i \sum_l \varepsilon_{jkl} \sigma_l.$$

For $u, v \in \mathbb{R}^3$, we have

$$\begin{aligned} (u \cdot \sigma)(v \cdot \sigma) &= (u_1 \sigma_1 + u_2 \sigma_2 + u_3 \sigma_3)(v_1 \sigma_1 + v_2 \sigma_2 + v_3 \sigma_3) \\ &= (u_1 v_1 + u_2 v_2 + u_3 v_3) I + u_1 v_2 \sigma_1 \sigma_2 + u_1 v_3 \sigma_1 \sigma_3 \\ &\quad + u_2 v_1 \sigma_2 \sigma_1 + u_2 v_3 \sigma_2 \sigma_3 + u_3 v_1 \sigma_3 \sigma_1 + u_3 v_2 \sigma_3 \sigma_2 \\ &= (u \cdot v) I + i (u_1 v_2 \sigma_3 - u_1 v_3 \sigma_2 - u_2 v_1 \sigma_3 + u_2 v_3 \sigma_1 \\ &\quad + u_3 v_1 \sigma_2 - u_3 v_2 \sigma_1) \\ &= (u \cdot v) I + i(u \times v) \cdot \sigma. \end{aligned}$$

$$\begin{aligned} \Rightarrow [u \cdot \sigma, v \cdot \sigma]_+ &= (u \cdot v) I + i(u \times v) \cdot \sigma + (u \cdot v) I + i(v \times u) \cdot \sigma \\ &= 2(u \cdot v) I \end{aligned}$$

$$\begin{aligned} [u \cdot \sigma, v \cdot \sigma] &= (u \cdot v) I + i(u \times v) \cdot \sigma - (u \cdot v) I - i(v \times u) \cdot \sigma \\ &= 2i(u \times v) \cdot \sigma. \end{aligned}$$

$$u = (1, 0, 0)$$

$$v = (0, 1, 0)$$

$$[\sigma_1, \sigma_2] = 2i(0, 0, 1) \cdot \sigma = 2i\sigma_3$$

"Skew-adjoint generators and the tangent space at the Identity."

$$\text{Set } X_j = -\frac{i}{2} \sigma_j.$$

$$\begin{aligned} \Rightarrow X_j^\dagger &= -X_j, \quad \text{tr}(X_j) = 0, \quad [X_j, X_k] = \left(-\frac{i}{2}\right)^2 [\sigma_j, \sigma_k] \\ &= -\frac{2i}{4} \sum_l \varepsilon_{jkl} \sigma_l \\ &= \sum_l \varepsilon_{jkl} X_l. \end{aligned}$$

Lemma: The tangent space at the

identity of $SU(2)$ is $\mathfrak{su}(2) = \{X : X^\dagger = -X, \text{tr}(X) = 0\}$

$$= \text{span}_{\mathbb{R}} \{X_1, X_2, X_3\}$$

Proof: If $U(t) \in SU(2)$, $U(0) = I$, and $X = U'(0)$, $\Rightarrow X^\dagger + X = 0$.

differentiating $U(t)^\dagger U(t) = I \Rightarrow (U'(0))^\dagger U(0) + U(0) U'(0) = 0$

$$(8) \quad U(t) = U(0) + t U'(0) + o(t) = I + tX + o(t).$$

$$\Rightarrow \det(U(t)) = \det(I + tX + o(t)) = 1 + t \operatorname{tr} X + o(t)$$

$$\begin{aligned} \frac{d}{dt} \det(U(t)) &= \det(U(t)) \operatorname{tr}(U(t)^{-1} \cdot U'(t)) \\ &= \operatorname{tr}(I \cdot X) = 0 \Rightarrow \operatorname{tr} X = 0. \end{aligned}$$

$$\cancel{U(t) = e^{itn \cdot \sigma} \text{ with } n \in S^2.}$$

$$\Leftarrow \text{ If } X^\dagger = -X, \operatorname{tr} X = 0 \Rightarrow X = \begin{pmatrix} ir & z \\ -\bar{z} & -ir \end{pmatrix} \begin{matrix} r \in \mathbb{R} \\ z = u + iv \in \mathbb{C} \end{matrix}$$

$$\begin{aligned} \Rightarrow X &= 2r X_3 + 2u X_2 + 2v X_1 \\ &= i(v \sigma_1 + u \sigma_2 + r \sigma_3) \end{aligned}$$

$$\Rightarrow e^{tX} = e^{it(u \cdot u \cdot v) \cdot \sigma} \in SU(2).$$

Dynamics of a two-state system.

time-independent Hamiltonian:

$$H = h_0 I + \mathbf{h} \cdot \boldsymbol{\sigma}, \quad h_0 \in \mathbb{R}, \quad \mathbf{h} = (h_1, h_2, h_3) \in \mathbb{R}^3.$$

the Schrödinger equation with $\hbar = 1$.

$$\frac{d}{dt} |\psi(t)\rangle = -i H |\psi(t)\rangle.$$

$$\Rightarrow |\psi(t)\rangle = e^{-itH} |\psi(0)\rangle \triangleq U(t) |\psi(0)\rangle.$$

$$\text{if } \|\psi(0)\rangle\| = 1 \Rightarrow \|\psi(t)\rangle\| = 1.$$

Lemma: Set $r = |\mathbf{h}|$. If $r > 0$, then.

$$U(t) = e^{-ih_0 t} \left[\cos(rt) I - i \sin(rt) \frac{\mathbf{h} \cdot \boldsymbol{\sigma}}{r} \right].$$

$$\text{If } r = 0 \Rightarrow U(t) = e^{-ih_0 t} I.$$

eg: $H = \hbar\omega I + \omega\sigma_3$, $\omega \in \mathbb{R}$ $|\psi(0)\rangle = |+\rangle$.

$$\begin{aligned} \Rightarrow |\psi(t)\rangle &= U(t) |\psi(0)\rangle \\ &= e^{i\hbar\omega t} \left[\cos(\omega t) I - i \sin(\omega t) \frac{\omega\sigma_3}{\omega} \right] |+\rangle \\ &= e^{i\hbar\omega t} \left[\cos(\omega t) I - i \sin(\omega t) \sigma_3 \right] |+\rangle \\ &= \underbrace{e^{i\hbar\omega t} \cos(\omega t)} |+\rangle - e^{i\hbar\omega t} i \sin(\omega t) |-\rangle \\ &= \frac{1}{\sqrt{2}} \left(e^{i(\hbar\omega t - \omega t)} |0\rangle + e^{i(\hbar\omega t + \omega t)} |1\rangle \right). \end{aligned}$$

$$\langle + | \psi(t) \rangle = e^{-i\hbar\omega t} \cos(\omega t) \quad \langle - | \psi(t) \rangle = -i e^{-i\hbar\omega t} \sin(\omega t).$$

$\downarrow$
 $\sigma_1 \rightarrow P = \cos^2(\omega t)$

eg Spin in a static magnetic field and the Zeeman Splitting

In the spin- $\frac{1}{2}$ realization (ignore the spatial motion)

$$H = k B \cdot \sigma \quad k = \frac{g e \hbar}{4 m c} \quad \hbar = 1$$

$\uparrow$ gyromagnetic ratio
 $\uparrow$ electronic charge
 $\downarrow$ mass $\rightarrow$ speed of light

eg: $B = (0, 0, B_3)$. $H = k B_3 \sigma_3$.

$$H|0\rangle = k B_3 |0\rangle \quad H|1\rangle = -k B_3 |1\rangle.$$

energy $\Rightarrow E|0\rangle = k B_3$ $E|1\rangle = -k B_3$

energy gap: $\Delta E = 2k|B_3|$

$$\frac{d}{dt} \langle \psi(t) | A | \psi(t) \rangle = i \langle \psi(t) | [H, A] | \psi(t) \rangle = 0 \quad (\Rightarrow [H, A] = 0).$$