

ch. 11 Fourier Analysis and the Free Particle

Last time: $p = -i\hbar \frac{d}{dq}$ ← lie abn grp of $\mathbb{R}$.

$$p e^{ikq} = \hbar k e^{ikq}$$

eigenvalues

eigenstates

$$\int_{-\infty}^{\infty} |e^{ikq}|^2 dq = \int_{-\infty}^{\infty} 1 dq$$

Want: decompose general state into these eigenstates

Two problems: 1. $e^{ikq} \notin L^2(\mathbb{R})$.

2. p ~~is not defined~~ on $L^2(\mathbb{R})$.
don't preserve

Two approaches:

(Fourier series)

①. Compactification. Just work on $\mathbb{R}$ with opposite sides identified. Consider function on $S^1 \cong \mathbb{R}/\mathbb{Z}$ symmetry

Nice basis: $e^{in\phi}$, $n \in \mathbb{Z}$.

Thm (Fourier series)

For $\psi \in L^2(S^1)$, $\psi(\phi) = \sum C_n e^{in\phi}$,

where

$$\hat{\psi}(n) = C_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\phi} \psi(\phi) d\phi$$

$$L^2(S^1) \leftrightarrow \ell^2(\mathbb{Z})$$

periodic func

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^N |C_n|^2 < \infty$$

$$\left(\lim_{N \rightarrow \infty} \sum_{n=-N}^N |C_n|^2 < \infty \right)$$

square-summable

For general L , rescale $\phi \rightarrow \phi_L = \frac{L}{2\pi} \phi$

$p = -i\hbar \frac{d}{d\phi_L}$ has eigenvalue $\frac{2\pi\hbar}{L} \cdot n$

$L \rightarrow \infty$, get continuous:

(2) Fourier Transform on $\mathbb{R}$

$$\psi(q) \rightarrow \tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(q) e^{-ikq} dq$$

cont. analogue of Fourier series.

Fourier inversion

$$\psi(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\psi}(k) e^{ikq} dk$$

Thm (Plancherel) Fourier transform $\mathcal{F}$ ~~and $\mathcal{F}^{-1}$~~ is

a unitary operator $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$.

In particular,

Ex: $\psi = \text{poly } e^{-q^2}$

$$\int_{-\infty}^{\infty} |\psi(q)|^2 dq = \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 dk.$$

First address the problem $\mathcal{F}$ not defined.

Restrict attention to "Schwartz function".

Def.

$$S(\mathbb{R}) = \left\{ \psi \in C^\infty(\mathbb{R}) : \sup_{q \in \mathbb{R}} \left| q^m \frac{d^n \psi}{dq^n} \right| < \infty \quad \forall m, n \geq 0 \right\}.$$

$S(\mathbb{R})$ is a function of $\mathbb{R}$, all derivatives exist. ψ and all derivs of ψ fall off faster than $\frac{1}{q^n}$ any $n \geq 0$.

Since a Schwartz function remains well behaved under arbitrary differentiation and multiplication, and Fourier transform exchange diff and multi.

$$\psi \in S(\mathbb{R}) \Leftrightarrow \tilde{\psi} \in S(\mathbb{R}).$$

still has problem $e^{ikq} \notin S(\mathbb{R})$.

Distribution -

Def a distribution is a cont. linear functional $T: S(\mathbb{R}) \rightarrow \mathbb{C}$.

$$\begin{array}{ccccc} S(\mathbb{R}) & \subset & L^2(\mathbb{R}) & \subset & S'(\mathbb{R}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{F} & & & & \\ \downarrow & & \downarrow & & \downarrow \\ S(\mathbb{R}) & \subset & L^2(\mathbb{R}) & \subset & S'(\mathbb{R}) \end{array}$$

$$e^{ikq} \text{ get interpreted as } \psi \in S(\mathbb{R}) \rightarrow \int_{-\infty}^{\infty} e^{ikq} \psi(q) dq = \tilde{\psi}(k) \\ \downarrow \\ = \langle e^{ikq}, \psi \rangle \in \mathbb{C}$$

claim: $\mathcal{F}: S'(\mathbb{R}) \rightarrow S'(\mathbb{R})$.

Proof Define $\langle \mathcal{F}T, \psi \rangle = \langle T, \mathcal{F}\psi \rangle$, $T \in S'(\mathbb{R})$, $\psi \in S(\mathbb{R})$.

Since $\mathcal{F}: S(\mathbb{R}) \rightarrow S(\mathbb{R})$ cont., $\mathcal{F}\psi \in S(\mathbb{R})$. Since T is a linear functional on $S(\mathbb{R})$, $\langle T, \mathcal{F}\psi \rangle \in \mathbb{C}$. Hence $\mathcal{F}T \in S'(\mathbb{R})$.

Use fourier transform to solve
Schrodinger eq.

$$i\hbar \frac{\partial}{\partial t} \psi(q,t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} \psi(q,t).$$

$$(\hbar=1)$$

Use $\frac{\partial \psi}{\partial q} = ik \hat{\psi}(k)$, SE becomes

(diff eq. $\Leftrightarrow$ multi op),

$$\frac{d}{dt} \hat{\psi}(k,t) = -\frac{1}{2m} (ik)^2 \hat{\psi}(k,t).$$

Sol $\hat{\psi}(k,t) = e^{-i\frac{k^2}{2m}t} \hat{\psi}(k,0)$

$$\Rightarrow \psi(q,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\frac{k^2}{2m}t} \hat{\psi}(k,0) e^{ikq} dk$$

Ch. 12 Position and the Free Particle

Def (Position operator)

$$Q\psi(q) = q\psi(q)$$

Similar problem as P.

Eigenfunctions of Q $\delta(q-a)$.

$$Q\delta(q-a) = a\delta(q-a).$$

Think of $\psi(q) = \int_{-\infty}^{\infty} \delta(q'-q) \phi(q') dq'$

as expanding in a basis $|q\rangle$.

Two basis for $\mathcal{H}$

Eigenfunction of P $\frac{1}{\sqrt{2\pi}} e^{ikq}$

Eigenfunction of Q $\delta(q-a)$

$|k\rangle$

$\in S'(\mathbb{R})$

Not $L^2(\mathbb{R})$

Can think of these as basis els, then.

$$\begin{aligned}\psi(q) &= \int_{-\infty}^{\infty} \delta(q-q') \psi(q') dq' \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{ikq} \tilde{\psi}(k) dk\end{aligned}$$

By Fourier transforms,

$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikq} \psi(q) dq$$

$$\psi = \int_{-\infty}^{+\infty} f(k-k') \tilde{\psi}(k') dk'.$$

On such functions, $\hat{P}\tilde{\psi}(k) = k\tilde{\psi}(k)$

$$Q = i\frac{d}{dk}$$

(multi and diff get interchanged !!).

Pirac notations. $|\psi\rangle \in \mathcal{H}$ ($\psi(q)$ or $\tilde{\psi}(k)$ are just in a basis)

dual space $\rightarrow \langle\psi| \in \mathcal{H}^*$, $\int \overline{\psi(q)} (\cdot) dq$
or $\int \overline{\tilde{\psi}(k)} (\cdot) dk$

$$\langle\psi_1|\psi_2\rangle = \langle\psi_1, \psi_2\rangle.$$

prop. ~~$[Q, H] = i\hbar$~~ $[Q, P] = i\hbar$

Proof: $[Q, P]\psi = QP\psi - PQ\psi$
 $= -i\hbar q\psi' + i\hbar \frac{d}{dq}(q\psi)$
 $= i\hbar\psi.$

Also, $[Q, H] \neq 0$ since ~~H~~ $H = \frac{P^2}{2m}$

$$[P, H] = 0.$$

Position unlike the momentum, is not preserved.

Heisenberg Uncertainty

Thm
$$\frac{\langle \psi | q^2 | \psi \rangle \langle \psi | p^2 | \psi \rangle}{\langle \psi | \psi \rangle \langle \psi | \psi \rangle} \geq \frac{1}{4}$$

Propagator (Position is not preserved under time evolution, we introduce propagator to describe how a state initially localized in pos evolves)

Sol to Schrodinger eq $|\psi(t)\rangle = e^{-i\hat{H}t} |\psi(0)\rangle$

In momentum representation

$$\tilde{\psi}(k,t) = e^{-i\frac{k^2}{2m}t} \tilde{\psi}(k,0)$$

What abt $\psi(q,t)$ in terms of $\psi(q,0)$?

Calculate by going to momentum space using Fourier transform back.

$$\psi(q,t) = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(q-q_0)} e^{-i\frac{k^2}{2m}t} dk \right) \psi(q_0,0) dq_0$$

↓
Propagator. $U(t, q-q_0)$

$$\cancel{U(t, q-q_0)} = \sqrt{\frac{m}{i2\pi t}} e^{-\frac{m}{i4t} (q-q_0)^2}$$

Note this is a distribution.

$$t \rightarrow z = t + i\epsilon$$