Ch 2. Rigid Body Motion.

Def 2.1 A mapping $g: \mathbb{R}^3 \to \mathbb{R}^3$ is a <u>rigid body transformation</u> if

① $\|g(p)-g(q)\| = \|p-q\|$, $\forall p, q \in \mathbb{R}^3$. ② $g_*(v \times w) = g_*(v) \times g_*(w)$ $\forall v, w \in \mathbb{R}^3$.

$g_*(v) = g(v) - g(0)$, $\|g_*(p) - g_*(q)\| = \|p-q\|$.

$g_*(0) = 0$.

$g_*(-v) = -g_*(v)$.

$\forall v_1, v_2 \in \mathbb{R}^3$, $v_1^T v_2 = \frac{1}{4}(\|v_1+v_2\|^2 - \|v_1 - v_2\|^2)$

$= \frac{1}{4}(\|g_*(v_1) + g_*(v_2)\| - \|g_*(v_1) - g_*(v_2)\|^2) = g_*(v_1)^T g_*(v_2)$.

* Rotation.

Space of rotation matrices $SO(n)$.

$SO(n) := \{R \in \mathbb{R}^{n\times n} : R^T R = I, \det R = 1\}$ → group.

$SO(3)$: rotation group of $\mathbb{R}^3$.

For $a \in \mathbb{R}^3$, define $(a)^\wedge = \hat{a} := \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$ ⇝ $\hat{a}^T = -\hat{a}$: skew-symmetric. then $a \times b = (a)^\wedge b$.

<u>Lemma 2.1</u> $R \in SO(3)$, $v, w \in \mathbb{R}^3$, then

$R(v \times w) = Rv \times Rw$, $R(w)^\wedge R^T = (Rw)^\wedge$.

$R = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$ $(Rv \times Rw)_1 = \begin{vmatrix} r_2 \cdot v & r_3 \cdot v \\ r_2 \cdot w & r_3 \cdot w \end{vmatrix} = (r_2 \times r_3)\cdot(v \times w) = r_1 \cdot (v \times w) = (R(v\times w))_1$.

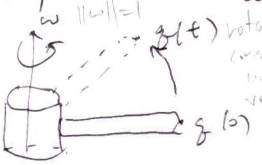
<u>Prop 2.2</u> $R \in SO(3)$ is a rigid body transformation.

∵ ~~$\|Rq - R$~~

$\|Rp - Rq\|^2 = (p-q)^T R^T R (p-q) = (p-q)^T(p-q) = \|p-q\|^2$.

* Exponential coordinates

g27



rotate with constant unit velocity

$\dot{q}(t) = \omega \times q(t) = \hat{\omega} q(t)$

$\Rightarrow q(t) = e^{\hat{\omega}t} q(0)$, $e^{\hat{\omega}t} = I + \hat{\omega}t + \frac{(\hat{\omega}t)^2}{2!} + \cdots$

$\hat{\omega}^T = -\hat{\omega}$, $so(n) := \{S \in \mathbb{R}^{n\times n} : S^T = -S\}$. Rotation $R(\omega, \theta) = e^{\hat{\omega}\theta}$

<u>Lemma 2.3</u> $\hat{a} \in so(3)$, then $\hat{a}^2 = aa^T - \|a\|^2 I$, $\hat{a}^3 = -\|a\|^2 \hat{a}$ (direct calculation)

If $\|\omega\| = 1$, then $e^{\hat{\omega}\theta} = I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\hat{\omega} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)\hat{\omega}^2$

$= I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$: Rodrigues' formula.

~~$a = \omega\theta$~~ ~~$\hat{a}^3 =$~~

$\hat{\omega}^3 = -\hat{\omega}$

Prop 2.4 $\hat{\omega} \in so(3)$, $\theta \in \mathbb{R} \Rightarrow e^{\hat{\omega}\theta} \in SO(3)$) WLOG, $\|\omega\| = 1$.

$(e^{\hat{\omega}\theta})^{-1} = e^{-\hat{\omega}\theta} = e^{\hat{\omega}^T\theta} = (e^{\hat{\omega}\theta})^T$.

$\because (I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta))(I - \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta))$

$= \underline{(I + \hat{\omega}^2(1-\cos\theta))^2} - \hat{\omega}^2\sin^2\theta = I$.

$I + 2\hat{\omega}^2(1-\cos\theta) + \hat{\omega}^4(1-2\cos\theta+\cos^2\theta) = I + 2\hat{\omega}^2(1-\cos\theta) - \hat{\omega}^2(1-2\cos\theta+\cos^2\theta)$

$\det e^{\hat{\omega}\theta} = 1$? $= I + \hat{\omega}^2 - \hat{\omega}^2\cos^2\theta$

$\theta = 0 \Rightarrow \det e^{\hat{\omega}\theta} = 1$, de to exp : continuous. □

Prop 2.5 $\forall R \in SO(3)$, $\exists \omega \in \mathbb{R}^3$, $\|\omega\| = 1$ & $\theta \in \mathbb{R}$ s.t. $R = \exp(\hat{\omega}\theta)$.

pf) $\det R = 1$. $\|Rv\| = \|v\| \Rightarrow |\lambda| = 1$ $\{1, e^{i\alpha}, e^{-i\alpha}\}$ or $\{-1, e^{i\alpha}, e^{-i\alpha}\}$.

$\therefore \{1, e^{i\alpha}, e^{-i\alpha}\}$ $1 \cdot e^{i\alpha} \cdot e^{-i\alpha} = 1$.

$\Rightarrow \mathrm{tr}R = \lambda_1 + \lambda_2 + \lambda_3$, $-1 \le \mathrm{tr}R \le 3$.

$\theta := \cos^{-1}\left(\frac{\mathrm{tr}R - 1}{2}\right)$. $\omega = \frac{1}{2\sin\theta}\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$ $\Rightarrow R = \exp(\hat{\omega}\theta)$ □

$\omega\theta \in \mathbb{R}^3$: exponential coordinates of R.

Thm 2.6 Any orientation $R \in SO(3)$ is equiv. to a rotation about a fixed axis $\omega \in \mathbb{R}^3$ through an angle $\theta \in [0, 2\pi)$.

Other representations ?

1. Euler angles

$(\alpha, \beta, \gamma) \to SO(3)$

rotate about z-axis by angle α,

" y-axis " β.

" z-axis " γ.

$R = R_z(\gamma) R_y(\beta) R_z(\alpha)$.

then $(\alpha, \beta, \gamma) \to SO(3)$ is surjective.

ZYZ, ZYX, YZX...

2. Quaternions.

$Q = q_0 + q_1 i + q_2 j + q_3 k$ " $(q_0, \vec{q})$, $\vec{q} = (q_1, q_2, q_3)$, $\|Q\|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$.

$R = \exp(\hat{\omega}\theta) \Rightarrow Q = (\cos(\frac{\theta}{2}), \omega\sin(\frac{\theta}{2})) \in \{Q : \|Q\| = 1\}$.

* Rigid Motion in $\mathbb{R}^3$.

Given $p \in \mathbb{R}^3$, $R \in SO(3)$, we can define $g(q) = p + Rq$: rigid motion.

$SE(3) := \{(p, R) : p \in \mathbb{R}^3, R \in SO(3)\} = \mathbb{R}^3 \times SO(3)$.

$SE(n) := \mathbb{R}^n \times SO(n)$

$g_*(v) = Rv$.

$\bar{g}$: homogeneous representation of g.

For $g \in (p, R) \in SE(3)$, define $\bar{g} := \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{4\times4}$. $\begin{pmatrix} R_1 & p_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} R_2 & p_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_1R_2 & R_1p_2+p_1 \\ 0 & 1 \end{pmatrix}$

Then $SE(3)$ forms a group : ① $g_1, g_2 \in SE(3) \Rightarrow g_1g_2 \in SE(3)$.

② $I \in SE(3)$

③ $\bar{g}^{-1} = \begin{pmatrix} R^T & -R^Tp \\ 0 & 1 \end{pmatrix} \in SE(3)$, $g^{-1} = (-R^Tp, R^T)$

Prop 2.7 $\forall g \in SE(3)$ is a rigid body transformation

$\because \|gq_1 - gq_2\| = \|Rq_1 - Rq_2\| = \|q_1 - q_2\|$, $g_*v \times g_*w = Rv \times Rw = R(v \times w)$.

*exponential coordinates.

$se(3) := \{(v, \hat{\omega}) : v \in \mathbb{R}^3, \hat{\omega} \in so(3)\}$. $\hat{\xi} \in se(3)$, $\overset{\text{twist}}{\hat{\xi}} = \begin{pmatrix} \hat{\omega} & v \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{4\times4}$.

$\vee, \wedge$: $\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}^\vee = \begin{bmatrix} v \\ w \end{bmatrix}$, $\begin{bmatrix} v \\ w \end{bmatrix}^\wedge = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$

Prop 2.8 $\forall \hat{\xi} \in se(3)$, $\theta \in \mathbb{R}$, $e^{\hat{\xi}\theta} \in SE(3)$.

pf) $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$. If $\omega = 0$, then $\hat{\xi}^2 = 0 \Rightarrow \exp(\hat{\xi}\theta) = I + \hat{\xi}\theta = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \in SE(3)$.

($\omega \neq 0$). Assume $\|\omega\| = 1$.

$\hat{\xi}' := g^{-1}\hat{\xi}g = \begin{bmatrix} I & -w\times v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & w\times v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \omega\omega^Tv \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & (\omega^Tv)\omega \\ 0 & 0 \end{bmatrix}$

$e^{\hat{\xi}'\theta} = \begin{pmatrix} e^{\hat{\omega}\theta} & (\omega^Tv)\omega\theta \\ 0 & 1 \end{pmatrix}$ $\because (\hat{\xi}')^2 = \begin{pmatrix} \hat{\omega}^2 & 0 \\ 0 & 0 \end{pmatrix}$, $(\hat{\xi}')^3 = \begin{pmatrix} \hat{\omega}^3 & 0 \\ 0 & 0 \end{pmatrix}$ ($\because \hat{\omega}\omega = \omega \times \omega = 0$)

$= I + \hat{\xi}'\theta + \frac{1}{2!}\hat{\xi}'^2\theta^2 + \cdots$

$\Rightarrow e^{\hat{\xi}\theta} = e^{g(\hat{\xi}'\theta)g^{-1}} = ge^{\hat{\xi}'\theta}g^{-1} = \begin{pmatrix} I & +\omega\times v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{\hat{\omega}\theta} & (\omega^Tv)\omega\theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & -\omega\times v \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} e^{\hat{\omega}\theta} & (\omega^Tv)\omega\theta + \omega\times v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & -\omega\times v \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\omega\times v) + \omega\omega^Tv\theta \\ 0 & 1 \end{pmatrix} \in SE(3)$

Prop 2.9 $\forall g \in SE(3), \exists \hat{\xi} \in se(3), \theta \in \mathbb{R}$ s.t. $g = e^{\hat{\xi}\theta}$. $\hat{\xi} = \begin{pmatrix} \hat{\omega} & v \\ 0 & 0 \end{pmatrix}$

f) i) $(R, p) = (I, 0)$: $\theta = 0$.

ii) $R = I, p \neq 0$. $\hat{\xi} := \begin{pmatrix} 0 & \frac{p}{\|p\|} \\ 0 & 0 \end{pmatrix}$, $\theta = \|p\|$. $\Rightarrow e^{\hat{\xi}\theta} = I + \hat{\xi}\theta = (I, p) = g$.

iii) $R \neq I$.

$$e^{\hat{\xi}\theta} = \begin{pmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{pmatrix} \overset{?}{=} \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix}$$

$\|\omega\| = 1$

Use Prop 2.5 to find $\hat{\omega}, \theta$ s.t. $e^{\hat{\omega}\theta} = R$

$A := (I - e^{\hat{\omega}\theta})\hat{\omega} + \omega\omega^T\theta$: nonsingular? $\langle\omega\rangle = \operatorname{span}\{\omega\}$

$\operatorname{Im}\hat{\omega} = \langle\omega\rangle^\perp$, $\ker\hat{\omega} = \langle\omega\rangle$. ~~$P_{N_\omega} = \omega\omega^T, P_{N_\omega}$~~ $P_{\langle\omega\rangle} = \omega\omega^T$, $P_{\langle\omega\rangle^\perp} = I - \omega\omega^T$

$\ker(I - e^{\hat{\omega}\theta}) = \langle\omega\rangle \Rightarrow I - e^{\hat{\omega}\theta} : \langle\omega\rangle^\perp \to \langle\omega\rangle^\perp$ is bijective.

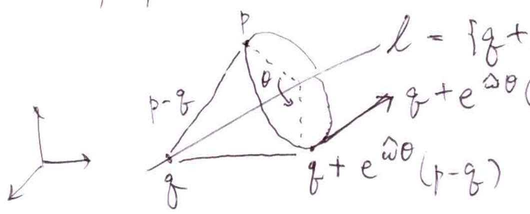
$\because e^{\hat{\omega}\theta} = \cos\theta I + (1-\cos\theta)\omega\omega^T + \sin\theta\hat{\omega}$: $\langle\omega\rangle^\perp \to \langle\omega\rangle^\perp$.

$A|_{\langle\omega\rangle} = \theta I$: invertible. $A|_{\langle\omega\rangle^\perp} = (I - e^{\hat{\omega}\theta})\hat{\omega}$: bijective on $\langle\omega\rangle^\perp$.

$A(\langle\omega\rangle) \subset \langle\omega\rangle$, $A(\langle\omega\rangle^\perp) \subset \langle\omega\rangle^\perp$, $A|_{\langle\omega\rangle}$, $A|_{\langle\omega\rangle^\perp}$: invertible. $\mathbb{R}^3 = \langle\omega\rangle \oplus \langle\omega\rangle^\perp$.

$\therefore A$ is invertible. $\square$

Def A <u>screw</u> (l, h, M) represents rotation by an amount $\theta = M$ about the axis l followed by translation by an amount $h\theta$ parallel to axis l. If $h = \infty$, then only pure translation by distance M.



$l = \{q + \lambda\omega : \lambda \in \mathbb{R}\}$, $\|\omega\| = 1$.

$gp = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega$,

$$g\begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$e^{\hat{\xi}\theta} = \begin{pmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\omega \times v) \times \omega\omega^T v\theta \\ 0 & 1 \end{pmatrix}$$

screw $\longleftrightarrow$ twist. $(\xi = (\omega, v), \theta)$

$(l, h, M) \Rightarrow \omega \Rightarrow v = -\omega \times q + h\omega$, $\hat{\xi} = \begin{pmatrix} \hat{\omega} & v \\ 0 & 0 \end{pmatrix}$ $\theta = M$

$l = \{q + \lambda\omega : \lambda \in \mathbb{R}\}$, $\|\omega\| = 1$

If $h = \infty$, $\hat{\xi} = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$

$$(\xi = (\omega, v), \theta) \to \begin{cases} h = \dfrac{\omega^T v}{\|\omega\|^2} \\ l = \begin{cases} \left\{\dfrac{\omega \times v}{\|\omega\|^2} + \lambda\omega : \lambda \in \mathbb{R}\right\} & \omega \neq 0 \\ \{0 + \lambda v : \lambda \in \mathbb{R}\} & \omega = 0 \end{cases} \\ M = \begin{cases} \|\omega\| & \omega \neq 0 \\ \|v\| & \omega = 0 \end{cases} \end{cases}$$

<u>Thm 2.11</u> Every rigid body motion can be realized by a rotation about an axis combined with a translation parallel to that axis.