Chapter 4. Robot Dynamics and Control

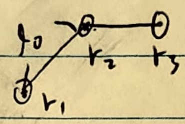
Motivation

Goal : Write the motion of the joints of robot by Lagrange's equation

$F_i = m_i \ddot{r}_i$ $\quad r_i$: position $\in \mathbb{R}^3$, $i = 1, \cdots, n$ (not linked)

$g_j(r_1, \cdots, r_n) = 0$ $\quad j = 1, \cdots, k$ $\quad$ k - constraint

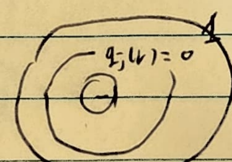
ex) $\| r_1 - r_2 \| = l_0$



$$F = \begin{bmatrix} m_1 I & & 0 \\ & \ddots & \\ 0 & & m_n I \end{bmatrix} \begin{bmatrix} \ddot{r}_1 \\ \vdots \\ \ddot{r}_n \end{bmatrix} + \sum_{j=1}^{k} \lambda_j \Gamma_j$$

(Lagrange multiplier)

$\Gamma_j \in \mathbb{R}^{3n}$ $\quad$ ex) $\Gamma_j = \nabla g_j$



3n+k equations with 3n+k variables.

- Computationally complex.

Consider $m = 3n - k$ variables. $q_1, \cdots, q_m$ (satisfying k - constraint)

$$r_i = f_i(q_1, \cdots, q_m) \iff g_j(r_1, \cdots, r_n) = 0$$

$i = 1, \cdots, n$ $\qquad j = 1, \cdots, k$

Def q_i : generalized coordinate.

$$L(q, \dot{q}) = \underbrace{T(q, \dot{q})}_{\text{kinetic E}} - \underbrace{V(q)}_{\text{Potential E}} \quad : \text{Lagrangian}$$

Thm Lagrange's equation

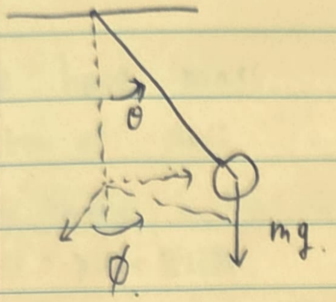
(minimizing path of L. $q(t) + \epsilon \eta(t)$)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \Upsilon_i \quad i = 1, \cdots, m$$

$\Updownarrow$

- $T = \frac{1}{2} \sum m_i \| \dot{r}_i \|^2$
- without constraint $F_i = m_i \ddot{r}_i$

Ex 4.1 Spherical pendulum.



generalized coordinate : $q = (\theta, \phi)$. (original r: 3 constraint 1.)

$$r(\theta,\phi) = \begin{bmatrix} l \sin\theta \cos\phi \\ l \sin\theta \sin\phi \\ -l\cos\phi \end{bmatrix}$$

$$\dot{r}(\theta,\phi) = \begin{bmatrix} l\cos\theta\cos\phi\,\dot\theta - l\sin\theta\sin\phi\,\dot\phi \\ l\cos\theta\sin\phi\,\dot\theta + l\sin\theta\cos\phi\,\dot\phi \\ l\sin\theta\,\dot\theta \end{bmatrix}$$

Kinetic energy

$$T = \frac{1}{2} m \|\dot r\|^2 = \frac{1}{2} m l^2 \left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2 \right)$$

Potential energy

$$V = -mgl\cos\theta$$

Lagrangian

$$L(q,\dot q) = \frac{1}{2} m l^2 \left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) + mgl\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \frac{d}{dt}\left(ml^2\dot\theta\right) = ml^2\ddot\theta$$

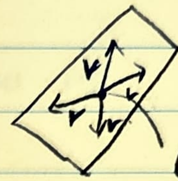
$$\frac{\partial L}{\partial \theta} = ml^2 \sin\theta\cos\theta\,\dot\phi^2 - mgl\sin\theta\,\dot\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = \frac{d}{dt}\left(ml^2\sin^2\theta\,\dot\phi\right) = ml^2\, 2\sin\theta\cos\theta\,\dot\theta\dot\phi + ml^2\sin^2\theta\,\ddot\phi$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = \begin{bmatrix} ml^2 & 0 \\ 0 & ml^2\sin^2\theta \end{bmatrix}\begin{bmatrix} \ddot\theta \\ \ddot\phi \end{bmatrix} + \begin{bmatrix} -ml\sin\theta\cos\theta\,\dot\phi^2 \\ 2ml^2\sin\theta\cos\theta\,\dot\theta\dot\phi \end{bmatrix} + \begin{bmatrix} mgl\sin\theta \\ 0 \end{bmatrix} = 0$$

2.2 Inertial properties of rigid bodies



COM

- Link have mass. $m = \int_V \rho(r)\, dV$
- Center of mass $\bar{r} = \frac{1}{m} \int_V \rho(r) r \, dV$

☆ Body frame : Fix center of mass at $(0,0,0)$

- $x(t) = p(t) + R(t) r$ $\quad \dot{x} = \dot{p} + \dot{R} r$

Kinetic E. $T = \frac{1}{2} \int_V \rho(r) \| \dot{p} + \dot{R} r \|^2 \, dV$

$$= \frac{1}{2} \int_V \rho(r) \left(\|\dot{p}\|^2 + \underbrace{2\dot{p}^T \dot{R} r}_{(ii)} + \underbrace{\|\dot{R} r\|^2}_{(iii)} \right) dV$$

(ii) $\quad \dot{p}^T \dot{R} \int_V \rho(r) r \, dV = 0 \quad (COM = 0)$

(iii) $\quad \frac{1}{2} \int_V \rho(r) (\dot{R} r)^T (\dot{R} r) \, dV = \frac{1}{2} \int_V \rho(r) (R \hat{w} r)^T (R \hat{w} r) \, dV \qquad (\dot{R} = R\hat{w})$

$$= \frac{1}{2} \int_V \rho(r) (\hat{w} r)^T R^T R (\hat{w} r) \, dV \qquad \left(\begin{aligned} \hat{w} r &= w \times r \\ &= -r \times w \\ &= -\hat{r} w \end{aligned} \right)$$

$$= \frac{1}{2} \int_V \rho(r) (-\hat{r} w)^T (-\hat{r} w) \, dV$$

$$= \frac{1}{2} w^T \int_V \rho(r) \hat{r}^T \hat{r} \, dV \, w$$

$\hat{r}^T \hat{r} = r r^T - \|r\|^2 I$

$$:= \frac{1}{2} w^T \underset{\uparrow}{I} w$$

$$I_{xx} = \int_V \rho(r) (y^2 + z^2) \, dr$$

(Inertia tensor)

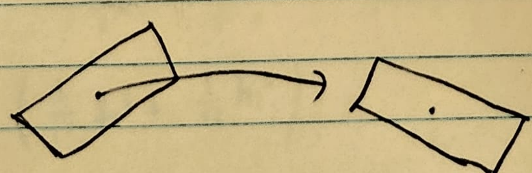
$$I_{xy} = -\int_V \rho(r)\, xy \, dr.$$

$w = (w_1, w_2, w_3)$

$$\hat{w} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}$$

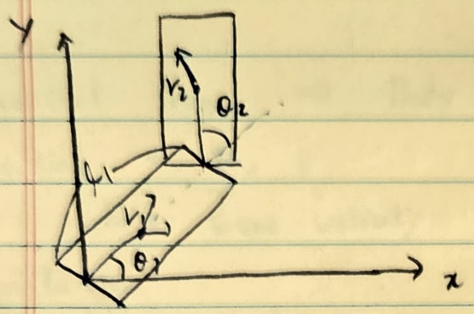
$$T = \frac{1}{2} m \|\dot{p}\|^2 + \frac{1}{2} w^T I w.$$

* Important : V : Fixed body frame even V is moving.



$$\int_V dV = \int_a^b \int_c^d dx dy$$

Ex) Two link planar robot



rectangular bar with mass m_i
(equaly distributed)

COM $(x_1, y_1) = (r_1 \cos\theta_1,\ r_1 \sin\theta_1)$

$(x_2, y_2) = (l_1 \cos\theta_1 + r_2 \cos(\theta_1+\theta_2),\ l_1 \sin\theta_1 + r_2 \sin(\theta_1+\theta_2))$

$v_1 = (\dot{x}_1, \dot{y}_1) = (-r_1 \sin\theta_1\, \dot{\theta}_1,\ r_1 \cos\theta_1\, \dot{\theta}_1)$

$v_2 = (\dot{x}_2, \dot{y}_2) = (-l_1 \sin\theta_1\, \dot{\theta}_1 - r_2 \sin(\theta_1+\theta_2)(\dot{\theta}_1 + \dot{\theta}_2),$
$\quad l_1 \cos\theta_1\, \dot{\theta}_1 + r_2 \cos(\theta_1+\theta_2)(\dot{\theta}_1 + \dot{\theta}_2))$

$w_1 = (0, 0, \dot{\theta}_1) \qquad w_2 = (0, 0, \dot{\theta}_1 + \dot{\theta}_2)$

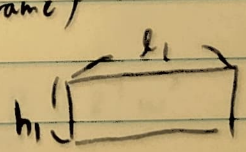
$T(\theta, \dot{\theta}) = \frac{1}{2} m_1 \|v_1\|^2 + \frac{1}{2} w_1^T I_1 w_1 + \frac{1}{2} m_2 \|v_2\|^2 + \frac{1}{2} w_2^T I_2 w_2$

$= \frac{1}{2} m_1 \|\dot{x}_1 + \dot{y}_1\|^2 + \frac{1}{2} (I_1)_{33} \dot{\theta}_1^2 + \frac{1}{2} m_2 \|\dot{x}_2 + \dot{y}_2\|^2 + \frac{1}{2} (I_2)_{33} (\dot{\theta}_1 + \dot{\theta}_2)^2$

$= \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}^T \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$

$= L(\theta, \dot{\theta})$ $\Rightarrow$ we can have Lagrange's equation

$(I_1)_{33} = \int_{V_1} \rho(r_1)(x_1^2 + y_1^2)\, dV_1$ (Body frame)



$\rho(r) = \frac{m_1}{l_1 h_1}$

$= \int_{V_1} \rho(r_1)\, r_1^2\, dV_1$

$= \int_{-\frac{h_1}{2}}^{\frac{h_1}{2}} \int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \frac{m_1}{l_1 h_1} (x_1^2 + y_1^2)\, dx_1\, dy_1$

$= m_1 \left(\frac{1}{12} l_1^2 + \frac{1}{12} h_1^2 \right)$

2.4 Newton - Euler equations.

Inertial frame $\Rightarrow$ Body frame.

inertial $\underline{f}$, $\underset{\text{Torque}}{\tau}$, $\underset{\text{velocity}}{\dot{p}}$ $\Rightarrow$ Body $f^b = R^T f$ $\quad v^b = R^T \dot{p}$

force

- Translation

$$\left(\begin{aligned} f = \tfrac{d}{dt}(m\dot{p}) = \tfrac{d}{dt}(mRv^b) &= Rm\dot{v}^b + \dot{R}mv^b \\ &= Rm\dot{v}^b + R\hat{w}^b m v^b \end{aligned} \right) \times R^T.$$

$$f^b = m\dot{v}^b + w^b \times (mv^b)$$

- Rotation

$I' = R I R^T$: inertia tensor of inertial frame $\left(\begin{aligned} I' &: \text{time depend} \\ I &: \text{constant} \end{aligned} \right)$

$$\begin{aligned} \tau &= \tfrac{d}{dt}(I' w^s) \\ &= I'\dot{w}^s + \dot{I}' w^s \\ &= I'\dot{w}^s + (\dot{R} I R^T + R I \dot{R}^T) w^s \qquad (\dot{R} = \hat{w}^s R) \\ &= I'\dot{w}^s + \left(\hat{w}^s R I R^T + R I (R^T(-\hat{w}^s))\right) w^s \\ &= I'\dot{w}^s + \hat{w}^s I' w^s - \underbrace{I'\hat{w}^s w^s}_{=0} \qquad \text{(Euler's equation)} \end{aligned}$$

$w^b = R^T w^s$, $\tau^b = R^T \tau$

$$\Rightarrow \begin{bmatrix} mI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{v}^b \\ \dot{w}^b \end{bmatrix} + \begin{bmatrix} w^b \times (mv^b) \\ w^b \times (I w^b) \end{bmatrix} = \begin{bmatrix} f^b \\ \tau^b \end{bmatrix}$$

(Newton - Euler equation)

$$\left(\begin{aligned} &\underline{R^T \hat{w}^s R}\, I R^T w^s \\ &\quad \| \\ &R^T w^s \times R = \widehat{R^T w^s} = \hat{w}^b \end{aligned} \right)$$

4.3. Dynamics of open chain Manipulator

4.3.1 Lagrangian for open chain robot.

Let $g_{sl_i}(0)$: COM of i th link

$$g_{sl_i}(\theta) = e^{\hat{\xi}_1 \theta_1} \cdots e^{\hat{\xi} \theta_i} g_{sl_i}(0)$$

Note

$g \in SE(3) = \left\{ g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \middle| \begin{array}{l} R \in SO(3) \\ p \in \mathbb{R}^3 \end{array} \right\}$,

$Ad_g = \begin{bmatrix} R & 0 \\ \hat{p}R & R \end{bmatrix}$

$\xi = \begin{pmatrix} v \\ w \end{pmatrix}$

$Ad_g \xi = \begin{bmatrix} Rv \\ \hat{p}Rv + Rw \end{bmatrix}$

Body velocity of COM of i th link

$$V^b_{sl_i} = J^b_{sl_i}(\theta)\,\dot{\theta}$$

$$J^b_{sl_i}(\theta) = [\,\xi_1^+, \cdots, \xi_i^+ \; 0 \cdots 0\,]$$

$$\xi_j^+ = Ad^{-1}_{(e^{\xi_j \theta_j}, \cdots, e^{\xi_i \theta_i} g_{sl_i}(0))} \xi_j \qquad (j \le i)$$

kinetic E of i th link.

$$T_i(\theta, \dot{\theta}) = \frac{1}{2} (V^b_{sl_i})^T \mathcal{M}_i V^b_{sl_i} = \frac{1}{2} \dot{\theta}^T J_i^T(\theta) \mathcal{M}_i J_i(\theta) \dot{\theta}$$

Total kinetic E.

$$T(\theta, \dot{\theta}) = \sum_{i=1}^{n} T_i(\theta, \dot{\theta}) = \frac{1}{2} \dot{\theta}^T \underset{\sim}{M(\theta)} \dot{\theta}$$

(different M)

$$M(\theta) := \sum_{i=1}^{n} J_i^T(\theta) \mathcal{M}_i J_i(\theta)$$

Potential E.

$$V(\theta) = \sum_{i=1}^{n} m_i g h_i(\theta)$$

Lagrangian

$$L(\theta, \dot{\theta}) = \frac{1}{2} \dot{\theta}^T M(\theta) \dot{\theta} - V(\theta)$$

4.3.2 Equation of motion

Write $L(\theta, \dot\theta) = \frac{1}{2} \sum_{i,j} M_{ij}(\theta)\, \dot\theta_i \dot\theta_j - V(\theta)$

Lagrange's eqn

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_i} - \frac{\partial L}{\partial \theta_i} = \Upsilon_i$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_i} = \frac{d}{dt}\left(\sum_j M_{ij}(\theta)\, \dot\theta_j\right) = \sum_j \left(M_{ij}\ddot\theta_j + \underbrace{\dot M_{ij}}_{\sum_{j,k} \frac{\partial \theta_k}{\partial t} \frac{\partial M_{ij}}{\partial \theta_k}} \dot\theta_j \right)$$

$$\frac{\partial L}{\partial \theta_i} = \frac{1}{2}\sum_{j,k} \frac{\partial M_{kj}}{\partial \theta_i} \dot\theta_k \dot\theta_j - \frac{\partial V}{\partial \theta_i}$$

$$\Rightarrow \sum_j M_{ij}\ddot\theta_j + \underbrace{\sum_{j,k}\left(\frac{\partial M_{ij}}{\partial \theta_k} \dot\theta_j \dot\theta_k - \frac{1}{2}\frac{\partial M_{kj}}{\partial \theta_i} \dot\theta_k \dot\theta_j \right)} + \frac{\partial V}{\partial \theta_i} = \Upsilon_i$$

$$\sum_{j,k}\left(\frac{1}{2}\frac{\partial M_{ij}}{\partial \theta_k} \dot\theta_j \dot\theta_k + \frac{1}{2}\frac{\partial M_{ik}}{\partial \theta_j} \dot\theta_j \dot\theta_k \right) \quad \text{symmetric } j \leftrightarrow k$$

$$\Gamma_{ijk} := \frac{1}{2}\left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i} \right) \quad \text{(Christoffel symbol)}$$

$\dot\theta_i \dot\theta_j$: Coriolis force.

$\dot\theta_i^2$: Centrifugal force.

Notation $C_{ij}(\theta, \dot\theta) = \sum_{k=1}^{n} \Gamma_{ijk} \dot\theta_k$ (Coriolis matrix)

Let $\Upsilon_i = \tau_i - N_i(\theta, \dot\theta)$

torque conservative force & friction

$$\Rightarrow M(\theta)\ddot\theta + C(\theta,\dot\theta)\dot\theta + N(\theta,\dot\theta) = \tau.$$

* $M(\theta)$: symetric positive definite

$\dot M - 2C$: skew symmetric