4. The Manipulator Jacobian.

Relationship between "Joint angles" and "End-effector ~~state~~"

4.1 End-effector velocity.

Recall "Joint space" Q. "Forward kinematics map" $g_{st} : Q \to SE(3)$. ($s \to$ base, $t \to$ tool.)

eg: $Q = S^1 \times S^1 \times \mathbb{R} \times S^1$ (S^1: revolute θ_1; $\mathbb{R}$: prismatic θ_3.) $\quad g = \begin{pmatrix} R & P \\ 0 & 1 \end{pmatrix}_{4\times4}$

$$\hat{V}_{st}^s = \frac{d}{dt}\left(g_{st}(\theta(t))\right) g_{st}^{-1}(\theta(t)) \overset{\text{chain rule}}{=} \sum_{i=1}^{n}\left(\frac{\partial}{\partial\theta_i} g_{st}\, \dot\theta_i\right) g_{st}^{-1} = \sum_{i=1}^{n}\left(\frac{\partial g_{st}}{\partial\theta_i} g_{st}^{-1}\right)\dot\theta_i$$

$$\Rightarrow V_{st}^s = J_{st}^s(\theta)\cdot\dot\theta. \text{ with } \underbrace{J_{st}^s}_{\in\mathbb{R}^{6\times n}} = \Big[\underbrace{\big(\underbrace{\tfrac{\partial g_{st}}{\partial\theta_1}}_{4\times4} \underbrace{g_{st}^{-1}}_{4\times4}\big)^\vee}_{6\times1}, \dots, \underbrace{\big(\tfrac{\partial g_{st}}{\partial\theta_n} g_{st}^{-1}\big)^\vee}_{6\times1} \Big]$$

velocity of joint i.

$$\boxed{\begin{aligned} &\text{Recall. } V = \begin{bmatrix} v \\ w \end{bmatrix} \quad \hat{V} = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} \\ &\begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix}^\vee = \begin{bmatrix} v \\ w \end{bmatrix}_{6\times1} \end{aligned}}$$

"Exponential formula" $g_{st}(\theta) = e^{\hat\xi_1\theta_1} \cdots e^{\hat\xi_n\theta_n} g_{st}(0)$ (unit twists) $g_{st}(0) \to$ initial configuration relate to the base frame

$$\Rightarrow \frac{\partial g_{st}}{\partial\theta_i} g_{st}^{-1} = e^{\hat\xi_1\theta_1}\cdots e^{\hat\xi_{i-1}\theta_{i-1}} \frac{\partial}{\partial\theta_i}\left(e^{\hat\xi_i\theta_i}\right)\cdots e^{\hat\xi_n\theta_n} g_{st}(0)\, g_{st}^{-1}$$

$$= e^{\hat\xi_1\theta_1}\cdots e^{\hat\xi_{i-1}\theta_{i-1}}(\hat\xi_i)\cdot e^{\hat\xi_i\theta_i}\cdots e^{\hat\xi_n\theta_n} g_{st}(0)\, g_{st}^{-1}(0)\, e^{-\hat\xi_n\theta_n}\cdots e^{-\hat\xi_1\theta_1}$$

$$= e^{\hat\xi_1\theta_1}\cdots e^{\hat\xi_{i-1}\theta_{i-1}}\,\hat\xi_i\, e^{-\hat\xi_{i-1}\theta_{i-1}}\cdots e^{-\hat\xi_1\theta_1}$$

$$\Rightarrow \left(\frac{\partial g_{st}}{\partial\theta_i} g_{st}^{-1}\right)^\vee = Ad_{\left(e^{\hat\xi_1\theta_1}\cdots e^{\hat\xi_{i-1}\theta_{i-1}}\right)}\,\xi_i$$

$$\Rightarrow \boxed{\begin{aligned} &J_{st}^s(\theta) = [\xi_1, \xi_2', \dots \xi_n'] \\ &\text{with } \xi_i' = Ad_{\left(e^{\hat\xi_1\theta_1}\cdots e^{\hat\xi_{i-1}\theta_{i-1}}\right)}\,\xi_i \end{aligned}}.$$

revolute joint

$$\xi_i = \begin{bmatrix} -w_i \times q_i \\ w_i \end{bmatrix}$$

prismatic joint

$$\xi_i = \begin{bmatrix} v_i \\ 0 \end{bmatrix}$$ ($v_i \to$ direction of translation)

Recall $V_{st}^b = g_{st}^{-1}\frac{d}{dt}\left(g_{st}(\theta(t))\right)$

$$\Rightarrow J_{st}^b = \left[\left(g_{st}^{-1}\frac{\partial g_{st}}{\partial\theta_i}\right)^\vee\right] = Ad^{-1}_{\left(e^{\hat\xi_i\theta_i}\cdots e^{\hat\xi_n\theta_n} g_{st}(0)\right)}\,\xi_i.$$

$$\Rightarrow J_{st}^s(\theta) = Ad_{(g_{st}(\theta))}\, J_{st}^b(\theta).$$

Recall $V_{st}^b = g_{st}^{-1}(\theta(t)) \frac{d}{dt}(g_{st}(\theta(t)))$

$$= \left[\left(g_{st}^{-1} \frac{\partial g_{st}}{\partial \theta_i}\right)^\vee\right] \dot{\theta}.$$

$\Rightarrow$ Body manipulator Jacobian $J_{st}^b = \left(g_{st}^{-1} \frac{\partial g_{st}}{\partial \theta_i}\right)^\vee$.

By exponentials formula.

$$g_{st}^{-1} \frac{\partial g_{st}}{\partial \theta_i} = g_{st}^{-1}(0) \cdot e^{-\hat{\xi}_n \theta_n} \cdots \cancel{e^{-\hat{\xi}_1 \theta_1} e^{\hat{\xi}_1 \theta_1}} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}} \hat{\xi}_i e^{\hat{\xi}_i \theta_i} \cdots e^{\hat{\xi}_n \theta_n} g_{st}(\theta)$$

$$= g_{st}^{-1}(0) \cdot e^{-\hat{\xi}_n \theta_n} \cdots e^{-\hat{\xi}_i \theta_i} \hat{\xi}_i e^{\hat{\xi}_i \theta_i} \cdots e^{\hat{\xi}_n \theta_n} g_{st}(0).$$

$$\left(g_{st}^{-1} \frac{\partial g_{st}}{\partial \theta_i}\right)^\vee = Ad_{(g_{st}^{-1}(0) e^{-\hat{\xi}_n \theta_n} \cdots e^{-\hat{\xi}_i \theta_i})} \xi_i$$

$$J_{st}^b \quad = Ad^{-1}_{(e^{\hat{\xi}_i \theta_i} \cdots e^{\hat{\xi}_n \theta_n} g_{st}(0))} \xi_i.$$

$$\Rightarrow \quad J_{st}^s(\theta) = Ad_{(g_{st}(\theta))} J_{st}^b(\theta).$$

The instantaneous velocity of a point.

①. q^b — point on the end-effector. (in body (tool) coordinates).

$$V_q^b = \hat{V}_{st}^b q^b = (J_{st}^b(\theta)\dot{\theta})^\wedge q^b.$$

② q^s — point ..... end-effector (in spatial (base frame)).

$$V_q^s = \hat{V}_{st}^s q^s = (J_{st}^s(\theta)\dot{\theta})^\wedge q^s.$$

$\bar{q}^s = g_{st}(\theta)\bar{q}^b \qquad \bar{q}^b = \begin{bmatrix} q^b \\ 1 \end{bmatrix}. \quad \bar{q}^s = \begin{bmatrix} q^s \\ 1 \end{bmatrix}$

If $q^b = 0$. $q^s = P(\theta)$ ~~$= g_{st}(\theta) q$~~ — the position component of the forward kinematics map.

$\begin{bmatrix} q^s \\ 1 \end{bmatrix} = \begin{bmatrix} R(\theta) & P(\theta) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q^b \\ 1 \end{bmatrix}$

$R_{st} = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$. $\Rightarrow$ $q^s = R(\theta) \underbrace{q^b}_{=0} + P(\theta)$

$= P(\theta)$

$\Rightarrow V_q^s = \begin{bmatrix} \dot{P}(\theta) \\ 0 \end{bmatrix}$ ~~$= R_{st} \hat{V}_{st}^b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$~~

$\hat{V}_{st}^b = \begin{bmatrix} \hat{\omega}^b & v^b \\ 0 & 0 \end{bmatrix}$

$= \hat{V}_{st}^s \begin{bmatrix} P(\theta) \\ 1 \end{bmatrix}$ ~~$= R_{st} \hat{V}_{st}^b \begin{bmatrix} P(\theta) \\ 1 \end{bmatrix}$~~

$\bar{q}^b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. $\quad \hat{V}_{st}^b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \hat{\omega}^b \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + v^b \\ 0 \end{bmatrix}$

$= \begin{bmatrix} v^b \\ 0 \end{bmatrix}$

$= \cancel{R_{st} \hat{V}_{st}^b R_{st}^{-1}}\; g_{st} \hat{V}_{st}^b g_{st}^{-1} \begin{bmatrix} P(\theta) \\ 1 \end{bmatrix}$

"The instantaneous velocity of a point".

① q^b — point on the end-effector (tool coordinates)

$$V_q^b = \hat{V}_{st}^b q^b = (J_{st}^b(\theta)\dot{\theta})^\wedge q^b .$$

② q^s — point on the end-effector (base ~~fr~~ coordinates)

$$V_q^s = \hat{V}_{st}^s q^s = (J_{st}^s(\theta)\dot{\theta})^\wedge q^s .$$

If $q^b = 0$, (origin of the tool coordinate)

$p(\theta) = q^s$ (the position component)

$$g_{st}(\theta) = \begin{pmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{bmatrix} q^s \\ 1 \end{bmatrix} = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q^b \\ 1 \end{bmatrix} = \begin{bmatrix} R(\theta) q^b + p(\theta) \\ 1 \end{bmatrix}$$

$$\Rightarrow V_q^s = \frac{d}{dt}\begin{bmatrix} q^s \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{p}(\theta) \\ 0 \end{bmatrix}$$

$$g_{st}^{-1}\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Rightarrow q^s = R(\theta) q^b + p(\theta) = p(\theta).$$

$$\Rightarrow g_{st}^{-1}\begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \hat{V}_{st}^s q^s = \hat{V}_{st}^s \begin{bmatrix} p(\theta) \\ 1 \end{bmatrix}$$

$$= g_{st} \hat{V}_{st}^b g_{st}^{-1} \begin{bmatrix} p(\theta) \\ 1 \end{bmatrix} = \underbrace{g_{st}}_{R_{st}} \hat{V}_{st}^b \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Equivalent expressions for the velocity of the origin of the tool frame.

⇒ "Relationship between joint velocity and end-effector velocity":

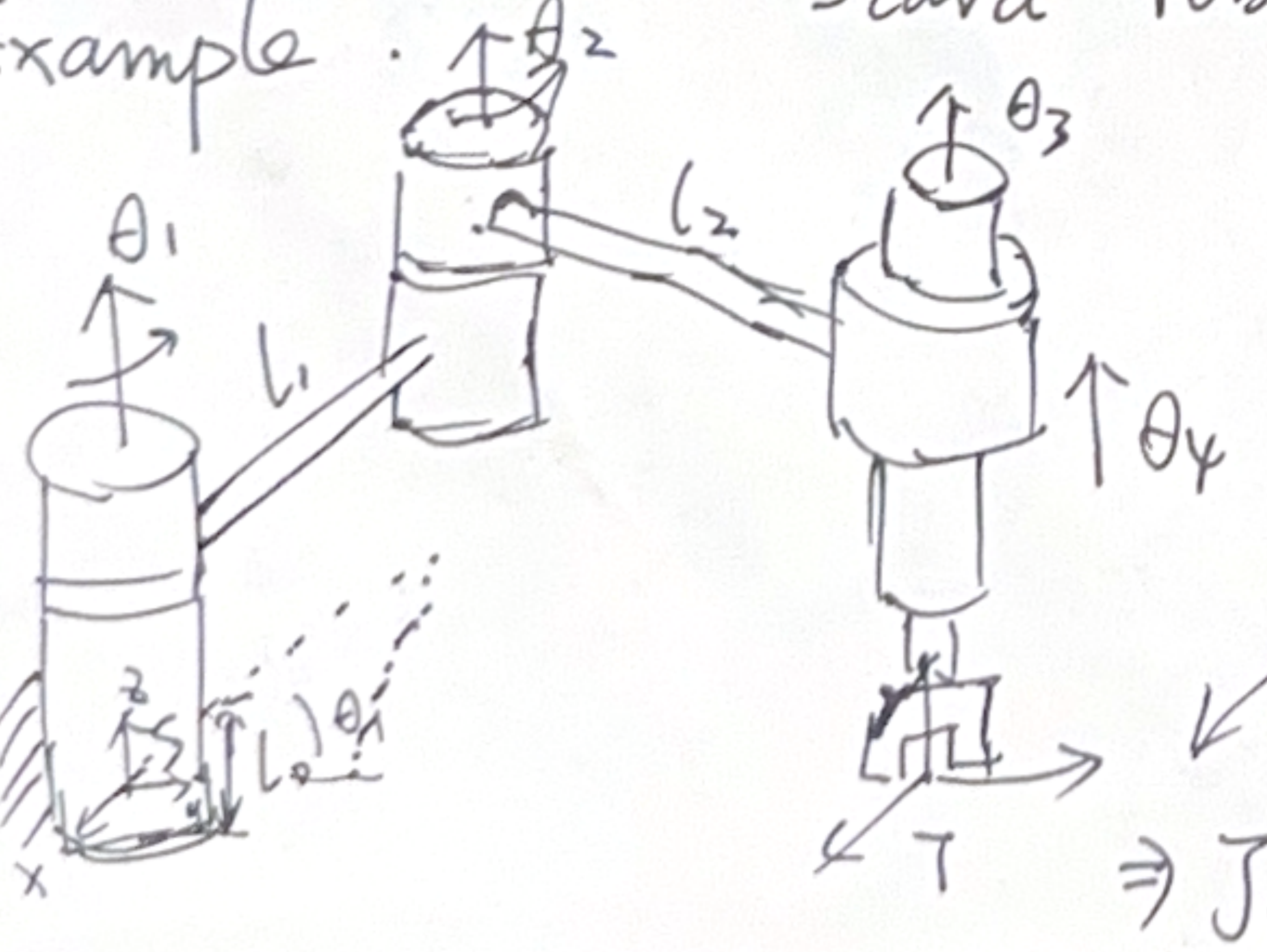
$$V_{st}^s(t) = J_{st}^s(\theta)\dot{\theta}(t)$$

if J_{st} is invertible. ⇒ $\dot{\theta}(t) = (J_{st}^s(\theta))^{-1} V_{st}^s(t)$

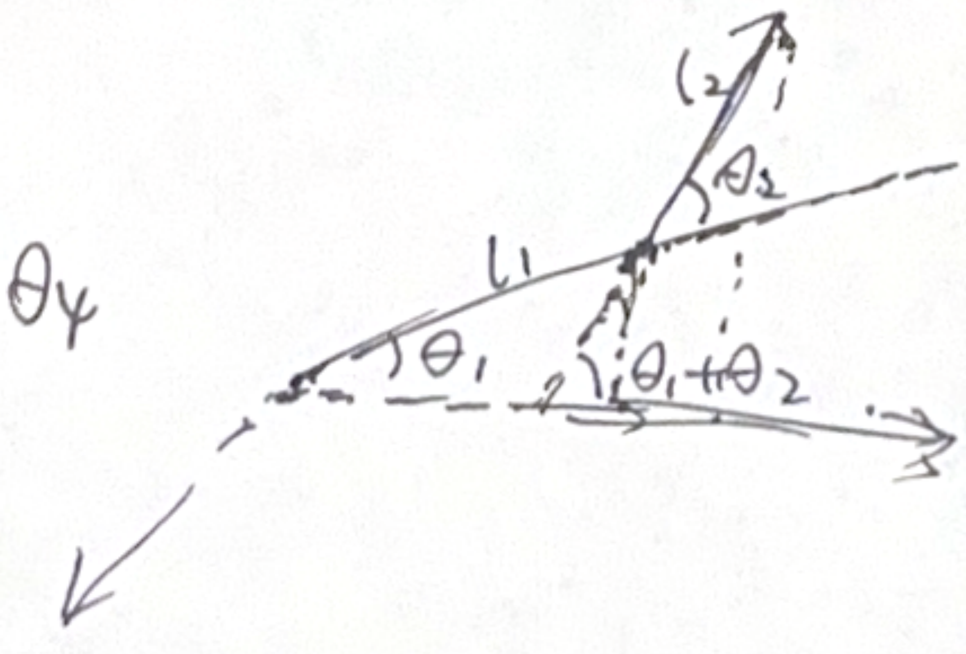
"singular configuration".

$$\xi = \begin{pmatrix} -q \times w \\ w \end{pmatrix}$$

Example. "Scara robot"



$w = (0, 0, 1)$. $q_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \xi_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$



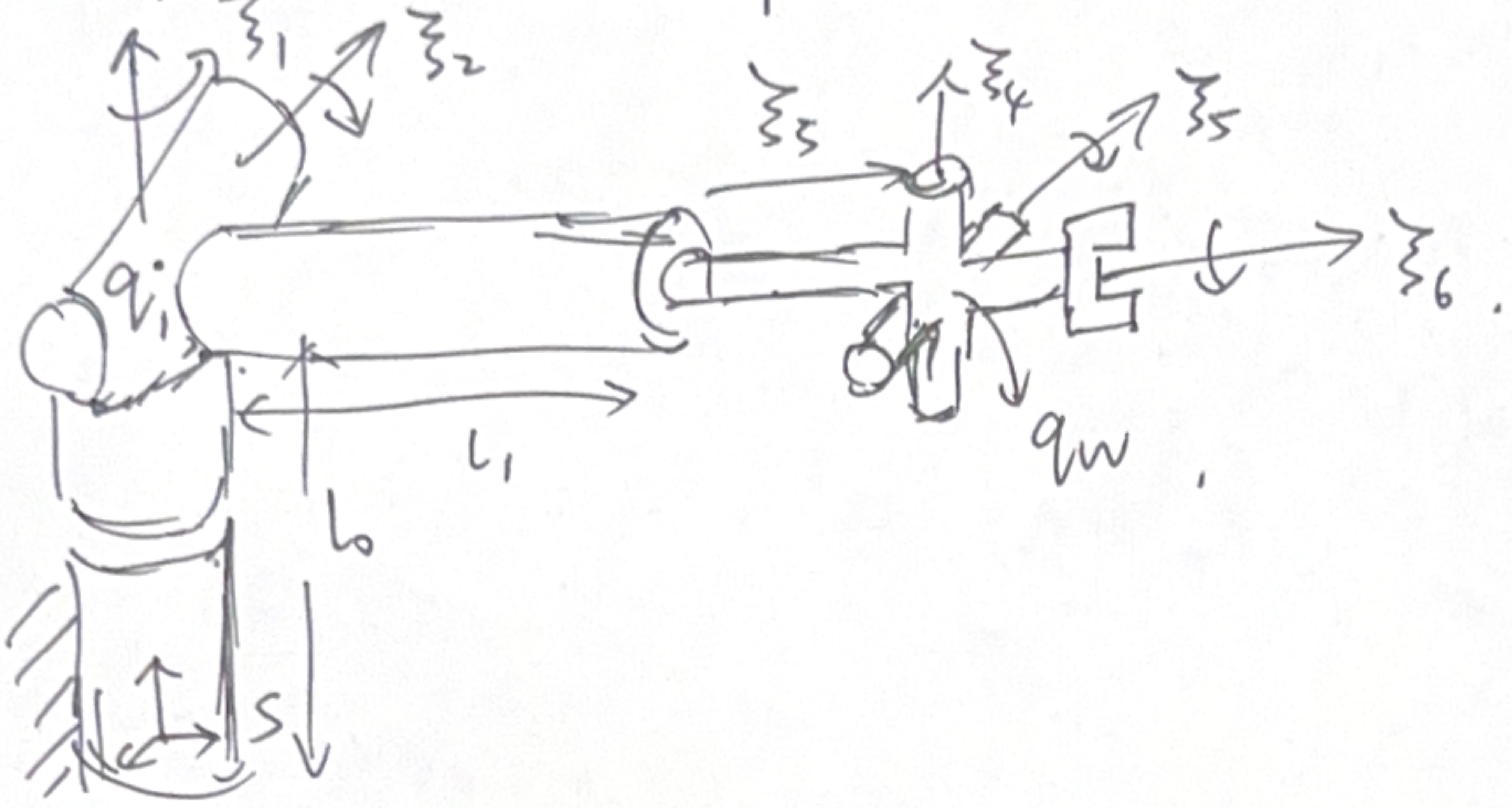
$$q_2 = \begin{pmatrix} -l_1\sin\theta_1 \\ l_1\cos\theta_1 \\ 0 \end{pmatrix} \Rightarrow \xi_2' = \begin{pmatrix} l_1\cos\theta_1 \\ l_1\sin\theta_1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

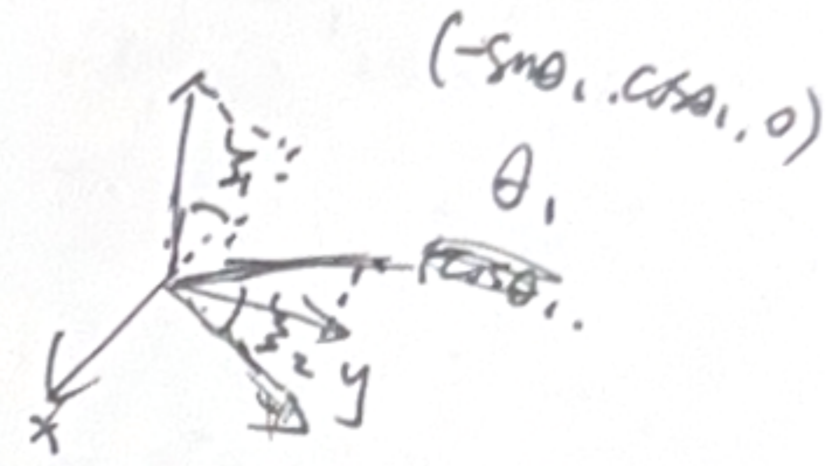
$$q_3 = \begin{pmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ 0 \end{pmatrix} \Rightarrow \xi_3' = \begin{pmatrix} \\ \end{pmatrix}$$

$$\Rightarrow J_{st}^s = [\xi_1, \xi_2', \xi_3'].$$

$$p(\theta) = \begin{pmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) \\ 0 \end{pmatrix} \Rightarrow \dot{p}(\theta) = \begin{pmatrix} -l_1 \cos\theta_1 \cdot \dot\theta_1 - l_2 \cos(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2) \\ -l_1 \sin\theta_1 \cdot \dot\theta_1 - l_2 \sin(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2) \\ 0 \end{pmatrix}$$

Example : " Stanford arm ".





$$q_1 = q_2 = (0, 0, l_0), \quad w_1 = (0, 0, 1). \quad w_2' = (-\cos\theta_1, -\sin\theta_1, 0).$$

$$\Rightarrow \xi_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \xi_2' = \begin{bmatrix} l_0 \sin\theta_1 \\ -l_0 \cos\theta_1 \\ 0 \\ -\cos\theta_1 \\ -\sin\theta_1 \\ 0 \end{bmatrix}.$$

$$v_3 = (0, 1, 0). \quad \xi_3' = \begin{pmatrix} e^{\hat{z}\theta_1} e^{-\hat{x}\theta_2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ 0 \end{pmatrix} = \begin{bmatrix} -\sin\theta_1 \cos\theta_2 \\ \cos\theta_1 \cos\theta_2 \\ -\sin\theta_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} v_3' \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$q_w' = l_1 \cdot v_3' + \theta_3 v_3' + q_1 = \begin{pmatrix} -(l_1+\theta_3) \sin\theta_1 \cos\theta_2 \\ (l_1+\theta_3) \cos\theta_1 \cos\theta_2 \\ l_0 - (l_1+\theta_3) \sin\theta_2 \end{pmatrix}.$$

$$w_4' = e^{\hat{z}\theta_1} e^{-\hat{x}\theta_2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}. \quad w_5' = e^{\hat{z}\theta_1} e^{-\hat{x}\theta_2} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$w_6' = e^{\hat{z}\theta_1} e^{-\hat{x}\theta_2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \xi_4' = \begin{pmatrix} -w_4' \times q_w' \\ w_4' \end{pmatrix} \quad \xi_5' = \begin{pmatrix} -w_5' \times q_w' \\ w_5' \end{pmatrix}$$

$$\xi_6' = \begin{pmatrix} -w_6' \times q_w' \\ w_6 \end{pmatrix}$$

$$\Rightarrow J_{st}^s = [\xi_1, \xi_2', \xi_3', \xi_4', \xi_5', \xi_6'].$$

4.2. End-effector Forces.

$$W = \int_{t_1}^{t_2} V_{st}^b \cdot F_t \, dt = \int_{t_1}^{t_2} \dot\theta \cdot \tau \, dt .$$

F_t → body wrench.

$$\Rightarrow \dot\theta \cdot \tau = V_{st}^b \cdot F_t = J_{st}^b \dot\theta \cdot F_t$$

$$\Rightarrow \dot\theta^T \tau = \dot\theta^T (J_{st}^b)^T F_t \Rightarrow \underline{\tau} = (J_{st}^b)^T F_t .$$

τ: joint torques.

Similar. $\tau = (J_{st}^s)^T F_s$.

Example: "Scara ~~robot~~".

$$(J_{st}^s)^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ l_1 c_1 & l_1 s_1 & 0 & 0 & 0 & 1 \\ l_1 c_1 + l_2 c_{12} & l_1 s_1 + l_2 s_{12} & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$\Rightarrow F_{N_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \quad F_{N_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{s.t.} \quad \tau = 0 .$$

4.4 Manipulability.

① ability to reach a certain position. — range of $p(\theta)$.

② ability to change the position. — ~~flexibility~~ velocity.

$\mu_1(\theta) = \sigma_{min}(J(\theta))$. $\quad \mu_2(\theta) = \dfrac{\sigma_{min}(J(\theta))}{\sigma_{max}(J(\theta))}$.

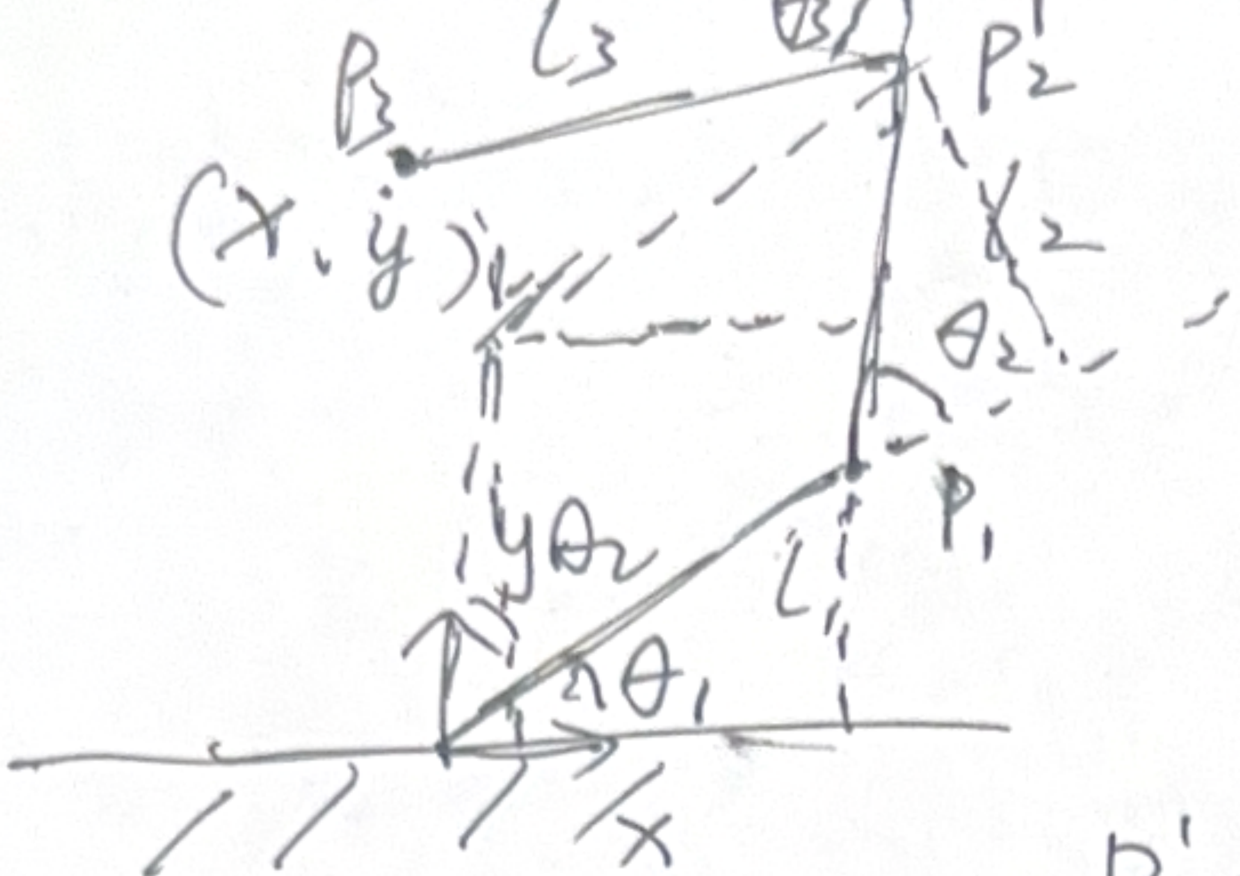
velocity. $\quad$ ↓ ~~sensitivity~~ average.

$\mu_3(\theta) = \det J(\theta)$.

volume of the velocity.

"Parallel Manipulators"

~~example 1.~~



$$P_1 = (l_1 \cos\theta_1, \ l_1 \sin\theta_1)$$

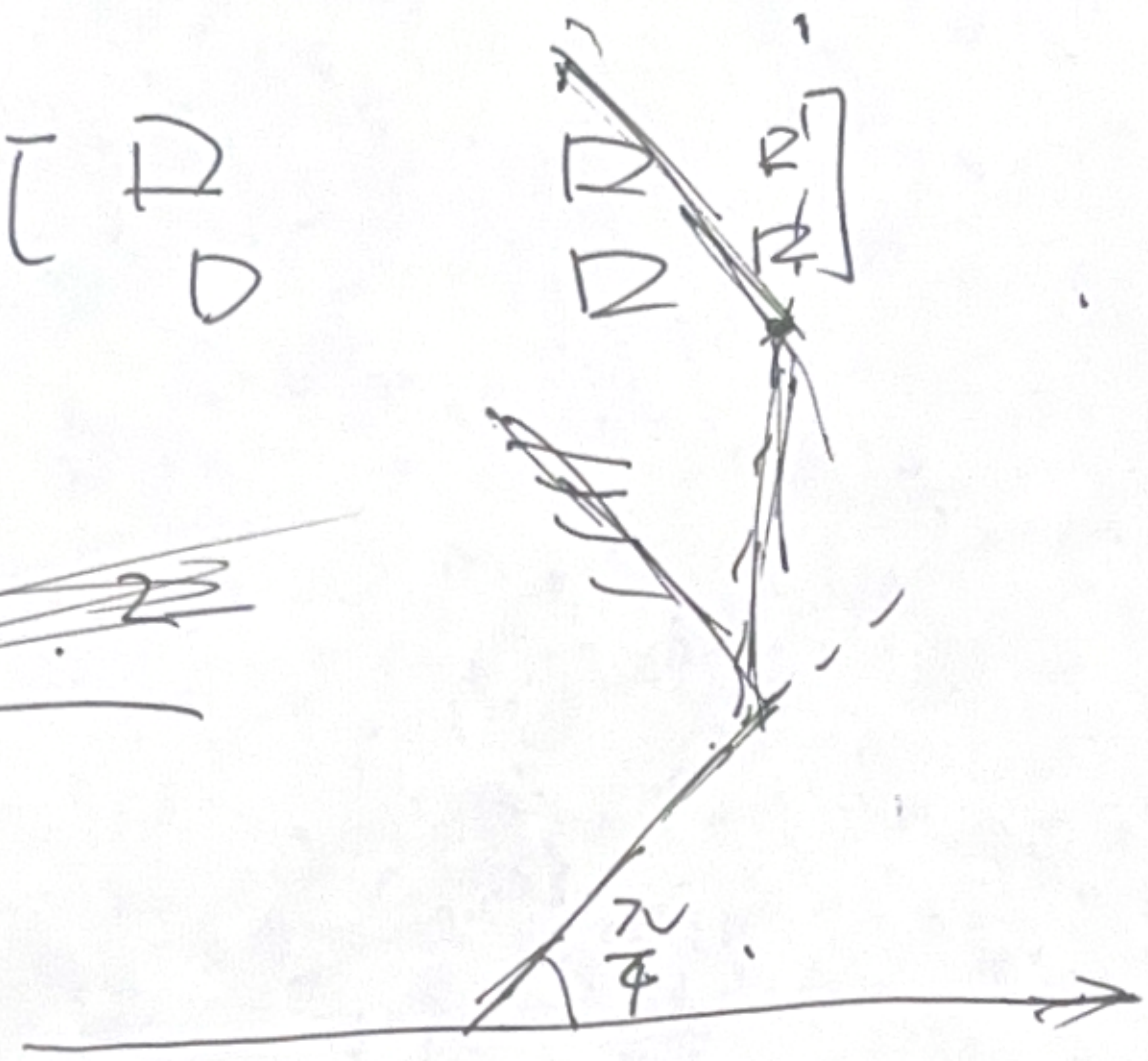
$$P_2' = (l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2), \ l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2))$$

$$P_3' = (l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3), \ l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) + l_3 \sin(\theta_1+\theta_2+\theta_3))$$

$$\frac{\partial P}{\partial \theta} = [$$

"planar manipulator".

~~Example 2~~



$$g_{st} = e^{\hat{\xi}_{11}\theta_{11}} \cdots e^{\hat{\xi}_{1n_1}\theta_{1n_1}} g_{st}(0) = e^{\hat{\xi}_{21}\theta_{21}} \cdots e^{\hat{\xi}_{2n_2}\theta_{n_2}} g_{st}(0).$$

degrees of freedom.

$$F = 6N - \sum_{i=1}^{g} (6 - f_i). \qquad \mathbb{R}^3$$

$$F = 3N - \sum_{i=1}^{g} (3 - f_i) \qquad \mathbb{R}^2.$$

Example. "Four bar linkage.