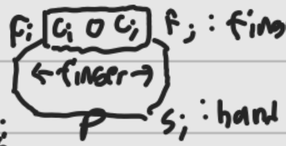
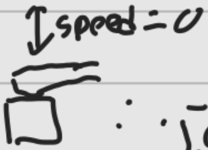
In this lecture, we will now consider the kinematics / statics of fingers
The velocity of contact points and finger points are equal

chapter 3, velocity of tool frame $V_{st}^{s} = J_{st}^{s}(\theta)\cdot\dot{\theta}$ body velocity $V_{st}^{b} = Ad^{-1}_{g_{st}(\theta)} V_{st}^{s}$
: instantaneous velocity 

f_i : finger point, C_i : contact point S, F, C : frame

$\therefore V^{b}_{s_i f_i} = Ad^{-1}_{g_{s_i f_i}(\theta_{f_i})} J^{s}_{s_i f_i}(\theta_{f_i})\dot{\theta}_{f_i}$, $J^{s}_{s_i f_i}$: Jacobian for F_i relative to S_i,
θ_{f_i} : vector of joint angles for finger i

Constraints ex) frictionless point contact : 

$\therefore [0\,0\,1\,0\,0\,0] V^{b}_{f_i c_i} = 0$
⇒ in general contact with wrench basis B_{c_i}, $B_{c_i}^{T} \cdot V^{b}_{f_i c_i} = 0$

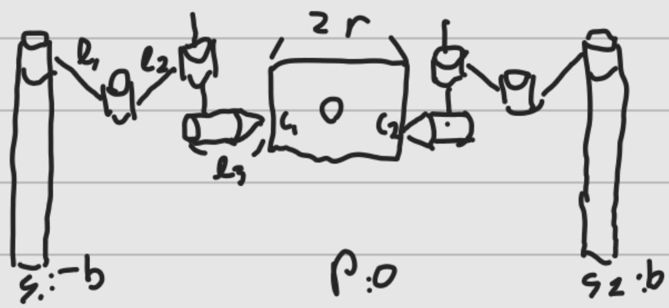
chapter 2 : $V^{b}_{f_i c_i} = Ad^{-1}_{g_{pc_i}} V^{b}_{f_i p} + V^{b}_{pc_i} = -Ad^{-1}_{g_{pc_i}} Ad_{g_{pf_i}} V^{b}_{pf_i} + V^{b}_{pc_i}$
$V^{b}_{ps_i} = 0 \Rightarrow V^{b}_{pf_i} = V^{b}_{s_i f_i}$, $V^{b}_{oc_i} = 0 \Rightarrow V^{b}_{pc_i} = Ad^{-1}_{g_{oc_i}} V^{b}_{po}$
$\therefore V^{b}_{f_i c_i} = -Ad^{-1}_{g_{pc_i}} Ad_{g_{pf_i}} V^{b}_{s_i f_i} + Ad^{-1}_{g_{oc_i}} V^{b}_{po}$

Combine with constraint ⇒ $B_{c_i}^{T} \left(Ad^{-1}_{g_{pc_i}} Ad_{g_{pf_i}} Ad^{-1}_{g_{s_i f_i}} \right) J^{s}_{s_i f_i}(\theta_{f_i})\dot{\theta}_{f_i} = B_{c_i}^{T} Ad^{-1}_{g_{oc_i}} V^{b}_{po}$
⇒ $B_{c_i}^{T} Ad^{-1}_{g_{s_i c_i}} J^{s}_{s_i f_i}(\theta_{f_i})\dot{\theta}_{f_i} = G_i^{T} V^{b}_{po}$ ($G_i = Ad^{-T}_{g_{oc_i}} B_{c_i}$)

hand Jacobian $J_h(\theta, x_o) = \mathrm{diag}\left(B_{c_i}^{T} Ad^{-1}_{g_{s_i c_i}} J^{s}_{s_i f_i}(\theta_{f_i}) \right)$, $x_o = g_{po}$
$J_h(\theta, x_o)\dot{\theta} = G^{T} V^{b}_{po}$ ($G = \begin{pmatrix} G_1 \\ \vdots \end{pmatrix}$) : fundamental grasping constraint
⇒ multifingered grasp is manipulable ⇔ $\mathcal{R}(G^{T}) \subset \mathcal{R}(J_h(\theta, x_o))$ (range space)

$\dot{\theta}_N \in \mathcal{N}(J_h)$: internal motion

Section 5.2, object wrench $F_o = G f_c$, joint torque $\tau = J_h^{T} f_c$ (f_c : contact force)

In the case of 2 fingers: 

Thus $G = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ r & 0 & 0 & 1 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & r & 0 & 0 & -r & 0 & 0 & 0 \end{bmatrix}$, $B_{c_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $J^s_{s_i f_i} = \begin{bmatrix} 0 & l_1 \cos\theta_1 & l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & 0 \\ 0 & l_1 \sin\theta_1 & l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$$g_{s_i c_i}^{-1} = g_{oc_i}^{-1} g_{po}^{-1} g_{s_i p}^{-1} \Rightarrow Ad_{g_{s_i c_i}}^{-1} = \underbrace{Ad_{g_{oc_i}}^{-1}}_{\text{constant}} \underbrace{Ad_{g_{po}}^{-1}}_{\text{depends on } x_n} \underbrace{Ad_{g_{s_i p}}^{-1}}_{\text{constant}}$$

wlog $R_{po} = I$, $p_{po} = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \Rightarrow R_{s_1 c_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $p_{s_1 c_1} = \begin{bmatrix} 0 \\ b-r \\ a \end{bmatrix}$, $R_{s_2 c_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $p_{s_2 c_2} = \begin{bmatrix} 0 \\ r-b \\ a \end{bmatrix}$,

$\therefore Ad_{g_{s_1 c_1}}^{-1} = \begin{bmatrix} R_{s_1 c_1}^T & \begin{matrix} b-r & 0 & 0 \\ 0 & a & r-b \\ -a & 0 & 0 \end{matrix} \\ 0 & R_{s_1 c_1}^T \end{bmatrix}$, $Ad_{g_{s_2 c_2}}^{-1} = \begin{bmatrix} R_{s_2 c_2}^T & \begin{matrix} 0 & a & b-r \\ r-b & 0 & 0 \\ a & 0 & 0 \end{matrix} \\ 0 & R_{s_2 c_2}^T \end{bmatrix}$ $J_n = \begin{bmatrix} J_{11} & 0 \\ 0 & J_{12} \end{bmatrix}$

$J_{11} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ r-b & l_1 c_1 + r - b & l_1 c_1 + l_2 c_{12} + r - b & 0 \\ 0 & l_1 s_1 & l_1 s_1 + l_2 s_{12} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $J_{22} = \begin{bmatrix} b-r & b-r+l_3 c_3 & b-r+l_3 c_3 + l_4 c_{34} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -l_3 s_3 & -l_3 s_3 - l_4 s_{34} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This case, grasp is not manipulable, $G^T \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \notin R(J_n)$

internal motions: $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

section 6: consider contact surfaces + rolling of objects against others

considers smooth surfaces

local coordinate chart $c: U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $(u,v) \mapsto x$

$\forall p \in S$, $\exists$ neighborhood $V \subset \mathbb{R}^3$, open $U \subset \mathbb{R}^2$, $c: U \to V \cap S$: differentiable, homeomorphism

$\forall \alpha \in U$, $\frac{\partial c}{\partial \alpha}(\alpha): \mathbb{R}^2 \to \mathbb{R}^3$ is injective $\Rightarrow$ S: regular

$\alpha = (u,v) \Rightarrow$ the tangent plane $= \text{span}\left(\frac{\partial c}{\partial u}, \frac{\partial c}{\partial v}\right)$

$\frac{\partial c}{\partial u} \cdot \frac{\partial c}{\partial v} = 0 \Rightarrow$ orthogonal chart : locally exists for $\forall$ regular surface

First fundamental form of $\mathbb{R}^3$ $I_p: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $\begin{bmatrix} c_u^T c_u & c_u^T c_v \\ c_v^T c_u & c_v^T c_v \end{bmatrix}$ $(p = c(u,v))$

positive definite $M_p: \mathbb{R}^2 \to \mathbb{R}^2$, $I_p = M_p \cdot M_p$, c: orthogonal $\Rightarrow M_p = \begin{bmatrix} \|c_u\| & 0 \\ 0 & \|c_v\| \end{bmatrix}$: metric tensor

Gauss map $N: S \to S^2$, $(u,v) \mapsto \frac{c_u \times c_v}{|c_u \times c_v|}$: normal unit of p

Second fundamental form : directional derivative of N

$II_p : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $\begin{bmatrix} c_u^T n_u & c_u^T n_v \\ c_v^T n_u & c_v^T n_v \end{bmatrix}$, $n_u = \frac{\partial n}{\partial u}$, $n_v = \frac{\partial n}{\partial v}$

Curvature tensor $K_p = M_p^{-T} II_p M_p^{-1} = \begin{pmatrix} c_u^T n_u / \|c_u\|^2 & c_u^T n_v / \|c_u\|\|c_v\| \\ c_v^T n_u / \|c_u\|\|c_v\| & c_v^T n_v / \|c_v\|^2 \end{pmatrix}$

$c(u,v)$: orthogonal $\Rightarrow$ normalized Gauss frame $[x\ y\ z] = \left[\frac{c_u}{\|c_u\|}\ \frac{c_v}{\|c_v\|}\ n\right]$

K_p: $[x\ y]^T \left[\frac{n_u}{\|c_u\|}\ \frac{n_v}{\|c_v\|}\right]$

Torsion form $T_p: \mathbb{R}^2 \to \mathbb{R}$, $(u,v) \mapsto \left[\frac{c_v^T c_{uu}}{\|c_u\|^2\|c_v\|}\ \frac{c_v^T c_{uv}}{\|c_u\|\|c_v\|^2}\right]$

curve $p(t) \in S$, contact frame $C(t)$: gauss frame of $p(t)$

if the frame of object O : fixed $\Rightarrow$ motion of C : $g_{oc}(t) \in SE(3)$

$p(t)$ lies in a single coordinate chart $c: U \to \mathbb{R}^3$, $\alpha(t) = c^{-1}(p(t))$

$V_{oc}^b = \left(\begin{bmatrix} M\dot{\alpha} \\ 0 \end{bmatrix}, \begin{bmatrix} \begin{matrix} 0 & -TM\dot{\alpha} \\ TM\dot{\alpha} & 0 \end{matrix} & KM\dot{\alpha} \\ -(KM\dot{\alpha})^T & 0 \end{bmatrix} \right)$ M: metric tensor, K: curvature tensor, T: torsion form of (c, U)

ex) $c(u,v) = \begin{pmatrix} \rho\cos u \cos v \\ \rho \cos u \sin v \\ \rho \sin u \end{pmatrix}$, $\{-\frac{\pi}{2} < u < \frac{\pi}{2}, -\pi < v < \pi\} \Rightarrow K = \begin{pmatrix} 1/\rho & 0 \\ 0 & 1/\rho \end{pmatrix}$, $M = \begin{pmatrix} \rho & 0 \\ 0 & \rho\cos u \end{pmatrix}$, $T = [0\ -\frac{\tan u}{\rho}]$

$p_o(t) \in S_o$, $p_f(t) \in S_f$: point of contact. $S_o, S_f \Rightarrow (c_o, U_o)$, (c_f, U_f) : charts

$\alpha_o = c_o^{-1}(p_o) \in U_o$, $\alpha_f = c_f^{-1}(p_f) \in U_f$: local coordinate, c_o, c_f : orthogonal

ψ: angle between $\frac{\partial c_f}{\partial u_f}, \frac{\partial c_o}{\partial u_o}$, $\eta = (\alpha_f, \alpha_o, \psi)$: contact coordinate

$W = \{\forall g_{of}\} \subset SE(3)$, C_o, C_f: coordinate frame coincide with normal Gauss frame at $p_o(t), p_f(t)$.

$L_o(t), L_f(t)$: coordinate frame relative to O, F $\quad L_o(t) = C_o$, $L_f(t) = C_f$

$v_{l_o l_f}$: body translational velocity, $w_{l_o l_f}$: body rotational velocity

if O, F contacts $\Rightarrow v_z = 0$, Pure rolling contact: $v = 0$, $w_z = 0$, pure sliding contact : $w = 0$, $v_z = 0$

$V_{blf} = \begin{bmatrix} v \\ w \end{bmatrix} = Ad_{g_{flf}}^{-1} V_{of} = \begin{bmatrix} R_{flf}^T v_{of} - R_{flf}^T \hat{p}_o w_{of} \\ R_{flf}^T w_{of} \end{bmatrix}$

orientation of x, y axis of C_f relative to x, y axis of C_o: $R_\psi = \begin{bmatrix} \cos\psi & -\sin\psi \\ -\sin\psi & -\cos\psi \end{bmatrix}$

Curvature of O at point of contact relative to x, y axis of C_f : $\tilde{K}_o = R_\psi K_o R_\psi$

$K_f + \tilde{K}_o$: relative curvature form

$$\left.\begin{aligned} \dot{\alpha}_f &= M_f^{-1} (K_f + \tilde{K}_o)^{-1} \left(\begin{bmatrix} -w_y \\ w_x \end{bmatrix} - \tilde{K}_o \begin{bmatrix} v_x \\ v_y \end{bmatrix} \right) \\ \dot{\alpha}_o &= M_o^{-1} R_\psi (K_f + \tilde{K}_o)^{-1} \left(\begin{bmatrix} -w_y \\ w_x \end{bmatrix} + K_f \begin{bmatrix} v_x \\ v_y \end{bmatrix} \right) \\ \dot{\psi} &= w_z + T_f M_f \dot{\alpha}_f + T_o M_o \dot{\alpha}_o \\ v_z &= 0 \end{aligned}\right) \text{motion of the contact coordinates } \dot{\eta}$$

ex) sphere rolling on a plane : $K_o = 0$, $K_f = \frac{1}{e} I$, $M_o = I$, $M_f = \begin{bmatrix} e & 0 \\ 0 & e\cos u_f \end{bmatrix}$

$T_o = 0$, $T_f = [0, -\frac{\tan u_f}{e}]$

$$\begin{pmatrix} \dot{u}_f \\ \dot{v}_f \\ \dot{u}_o \\ \dot{v}_o \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ \sec u_f \\ -e\sin\psi \\ -e\cos\psi \\ -\tan u_f \end{pmatrix} w_z + \begin{pmatrix} -1 \\ 0 \\ -e\cos\psi \\ e\sin\psi \\ 0 \end{pmatrix} w_y$$

Grasping with rolling

$J_h(\theta, x_o, \eta)\dot{\theta} = G^T(\eta) V_{po}^b$, $\dot{\eta}_i = A_i(\eta_i) V_{of_i}^b$ (η: contact coordinates,

$B_{c_i}^T V_{oc_i}^b = 0$: contact point cannot move in constrained directions

$K_f + R_\psi K_o R_\psi^T$ is full rank $\Rightarrow$ there is a smooth local bijection between η, g_{of_i}

$\Rightarrow J_h(\theta, x_o)\dot{\theta} = G^T(\theta, x_o) V_{po}^b$, if not

$Ad_{g_{s_i c_i}}^{-1} = Ad_{g_{oc_i}}^{-1} Ad_{g_{po}}^{-1} Ad_{g_{s_i p}}^{-1}$, $g_{oc_i} = \begin{bmatrix} \frac{\partial c_{o_i}}{\partial \alpha_{o_i}} M_{o_i}^{-1} & n_{o_i} & c_{o_i}(\alpha_i) \\ 0 & 0 & 1 \end{bmatrix}$