Chapter 2. Hilbert Spaces.

→ the fundamental mathematical framework for quantum mechanics.

1° quantum state spaces: complex vector spaces.

eg: $H_1 = \{ \alpha|0\rangle + \beta|1\rangle.\ \alpha.\beta \in \mathbb{C} \}$. $|0\rangle.\ |1\rangle$ — kets.

Basis : $B_1 = (|0\rangle.\ |1\rangle)$.

H_1 is the set of all formal linear combinations of the elements of B_n with complex coefficients.

More general: Basis : $B_n = (|\vec{b}\rangle)_{\vec{b}\in\{0,1\}^n}$.

① $H_n = \sum_{\vec{b}\in\{0,1\}^n} \mathbb{C}\,|\vec{b}\rangle$. $\dim(H_n) = 2^n$.

Since $\dim(H_n) = 2^n$. we can rewrite $B_n = (|0\rangle_n.\ |1\rangle_n.\ \cdots.\ |2^n-1\rangle_n)$
$\triangleq (|b\rangle_n)_{b\in\mathbb{Z}_{2^n}}$.

and ② $H_n = \{ \sum_{b=0}^{2^n-1} \alpha_b |b\rangle_n : \alpha_b \in \mathbb{C}$ with $b \in \mathbb{Z}_{2^n} \}$.

③ vector representations.

For k-dimension complex vector space H. basis $B \triangleq (|b_0\rangle.\ \cdots.\ |b_{k-1}\rangle)$

then each $|\varphi\rangle \in H$. we can write $|\varphi\rangle = \alpha_0|b_0\rangle + \cdots + \alpha_{k-1}|b_{k-1}\rangle$.

isomorphism $H \longleftrightarrow \mathbb{C}^k$.

$|\varphi\rangle \longleftrightarrow (\alpha_0.\ \cdots.\ \alpha_{k-1}) \triangleq |\varphi\rangle_B$.

$$\Rightarrow |\varphi\rangle = (|b_0\rangle.\ \cdots.\ |b_{k-1}\rangle)\begin{pmatrix}\alpha_0\\ \vdots \\ \alpha_{k-1}\end{pmatrix}.$$

eg: $|\varphi\rangle = |0\rangle + i|1\rangle$. $B = (|0\rangle.\ |1\rangle)$.

then $|\varphi\rangle_B = (1.\ i)$.

2°. Inner products.

$\langle \cdot | \cdot \rangle : H \times H \to \mathbb{C}$.

$(|\varphi\rangle, |\psi\rangle) \mapsto \langle \varphi | \psi \rangle$.

bra.

$\langle \varphi | \in H^*$.

$\langle \varphi | : H \to \mathbb{C}$.

$|\psi\rangle \mapsto \langle \varphi | \psi \rangle$.

Isomorphism.

$|\varphi\rangle = \sum_{i=0}^{m-1} \alpha_i |\varphi_i\rangle$, $|\psi\rangle = \sum_{i=0}^{m-1} \beta_i |\psi_i\rangle$.

then. $\langle \varphi | \psi \rangle = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \overline{\alpha_i} \beta_j \langle \varphi_i | \psi_j \rangle$.

complex conjugate of α_i

※ Inner product on H with respect to the basis $B = (|b_0\rangle, \cdots, |b_{k-1}\rangle)$.

$$\langle b_i | b_j \rangle_B = \delta_{ij} = \begin{cases} 1. & i=j \\ 0. & i \neq j \end{cases}.$$

then. for. $|\varphi\rangle = (|b_0\rangle, \cdots, |b_{k-1}\rangle) \cdot \underbrace{\begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{k-1} \end{pmatrix}}_{|\varphi\rangle_B}$. $|\psi\rangle = (|b_0\rangle, \cdots, |b_{k-1}\rangle) \underbrace{\begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{k-1} \end{pmatrix}}_{|\psi\rangle_B}$.

we have $\langle \varphi | \psi \rangle = (\overline{\alpha_0} \cdots \overline{\alpha_{k-1}}) \cdot \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{k-1} \end{pmatrix} = |\varphi_B\rangle^* \cdot |\psi\rangle_B$.

Bra-ket notation (Dirac notation). introduced by Paul Dirac 1930s.

Bra → dual vectors. ket → state vector.

$(H, \langle \cdot | \cdot \rangle)$ — Hilbert space.

Norm. $\| |\varphi\rangle \| \triangleq \|\varphi\| = \sqrt{\langle \varphi | \varphi \rangle}$.

⇓

Orthogonality:

※ $|\varphi\rangle$ and $|\psi\rangle$ are orthogonal to each other $\Leftrightarrow \langle \varphi | \psi \rangle = 0$.

※ orthonormal B. $\forall |\varphi\rangle, |\psi\rangle \in B. \Rightarrow \langle \varphi | \psi \rangle = 0$. $\|\varphi\| = \|\psi\| = 1$.

sequence

$B = (|b_0\rangle, \dots, |b_{k-1}\rangle)$ is a orthonormal basis of $(H, \langle\cdot|\cdot\rangle_B)$.

$|\varphi\rangle = \sum_{i=0}^{k-1} \langle b_i|\varphi\rangle |b_i\rangle$.

3°. Linear maps : $H \longrightarrow H'$.

$\mathbb{C}^k \quad \mathbb{C}^l$.

basis $\quad B \quad C$

$f \in \mathrm{Hom}(H, H') \longrightarrow \mathrm{Mat}_{B,C}(f) \in \mathbb{C}^{k\times l}$.

eg : Pauli X operator. on H_1 $\quad B = (|0\rangle, |1\rangle)$

$X : H_1 \to H_1 \qquad \alpha|0\rangle + \beta|1\rangle \longmapsto \beta|0\rangle + \alpha|1\rangle$.

Set. $\mathrm{Mat}_B(X) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. then. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$.

$\Rightarrow \begin{cases} a\alpha + b\beta = \beta \\ \alpha + d\beta = \alpha \end{cases} \Rightarrow \begin{cases} a=0 \\ b=1 \\ c=1 \\ d=0 \end{cases}$ ie. $\mathrm{Mat}_B(X) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

basis $C = (|x_+\rangle, |x_-\rangle) = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$.

$\Rightarrow \mathrm{Mat}_C(X) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. differet basic $\Rightarrow$ different Matrix representation

Pauli Z operator : $\mathrm{Mat}_B(Z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Pauli Y operator : $\mathrm{Mat}_B(Y) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Hadamard operator. $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Quantum gates.

Prop : $\langle\varphi|T|\psi\rangle = \langle\varphi| \circ T|\psi\rangle$. $\mathrm{Mat}_{B,C}(T) = (\langle c_i|T|b_j\rangle)_{i\in\mathbb{Z}_l, j\in\mathbb{Z}_k} \in \mathbb{C}^{l,k}$.

4°. The Hilbert-Schmidt inner product. ($\mathbb{C}^{l\times k}$)

$\langle \cdot | \cdot \rangle : \mathbb{C}^{(l\times k)} \times \mathbb{C}^{(l\times k)} \to \mathbb{C}$.

$(A, B) \mapsto \langle A | B \rangle = tr(A^* B)$. $\quad A^* = \overline{A}^T \in \mathbb{C}^{k,l}$

$A = (a_0, \cdots, a_{k-1}) \quad B = (b_0, \cdots, b_{k-1})$

$\Rightarrow A^* = \begin{pmatrix} a_0^* \\ \vdots \\ a_{k-1}^* \end{pmatrix} \Rightarrow A^*B = \begin{pmatrix} a_0^* b_0 & a_0^* b_1 & \cdots & a_0^* b_{k-1} \\ \vdots & & & \\ a_{k-1}^* b_0 & a_{k-1}^* b_1 & \cdots & a_{k-1}^* b_{k-1} \end{pmatrix}$

$\Rightarrow tr(A^*B) = \sum_{i=0}^{k-1} a_i^* b_i$.

5°. Endomorphism.

$\mathbb{C}^{k,k} : H \longrightarrow H$. $\quad A \in \mathbb{C}^{k,k} \Leftrightarrow A \in End(H)$.

eg: Pauli operators X. Y. Z. on H_1 with respect to $(|0\rangle, |1\rangle)$ basis.
are Pauli matrices.

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Hadamard operator on H_1 with respect to $(|0\rangle, |1\rangle)$.

$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. involutions.

P73. check properties of matrix $A \in \mathbb{C}^{(k,k)}$.

$A \in \mathbb{C}^{(k,k)}$ or $A \in End(H)$ is called an involution if $A^2 = I_k$. or $A^2 = I_H$.

6. Hermitian matrices or operators. $\Leftrightarrow A = A^*$

Unitary matrices or operators $\Leftrightarrow$ $U^*U = UU^* = I_k$. used to model the evolution of quantum systems over time.

Rows or columns ~~form~~ an orthonormal basis of $\mathbb{C}^k$.

eg: Pauli X.Y.Z. Hadamard operator H.

7°. Outer products.

$B = (|b_0\rangle, \cdots, |b_{k-1}\rangle)$ be an orthonormal basis of H.

$|\varphi\rangle\langle\psi| : H \to H$

$\in End(H)$ $\quad |\xi\rangle \mapsto |\varphi\rangle \cdot \underbrace{\langle\psi|\xi\rangle}_{\in \mathbb{C}}$

$$Mat_B(|\varphi\rangle\langle\psi|) = (\alpha_i \overline{\beta_j}) = \begin{pmatrix} \alpha_1\overline{\beta_1} & \cdots & \alpha_1\overline{\beta_k} \\ \vdots & & \vdots \\ \alpha_k\overline{\beta_1} & & \alpha_k\overline{\beta_k} \end{pmatrix}$$

density operators

ρ_{123}

eg: H_1 single-qubit state space with Basis $(|0\rangle, |1\rangle)$.

Example of outer product of kets in H_1 are $|0\rangle\langle 0|$, $|0\rangle\langle 1|$, $|1\rangle\langle 0|$, $|1\rangle\langle 1|$.

For $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle \in H$, then

$$|0\rangle\langle 0| (\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle + \beta\langle 0|1\rangle = \alpha|0\rangle.$$

$$|0\rangle\langle 1| (\alpha|0\rangle + \beta|1\rangle) = \beta|0\rangle$$

$$|1\rangle\langle 0| (\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle$$

$$|1\rangle\langle 1| (\alpha|0\rangle + \beta|1\rangle) = \beta|1\rangle.$$

8° Projections. — measurement.

Def: $P \in End(H)$ is called a projection if $P^2 = P$.

A projection $P \in End(H)$ is called orthogonal if

$P|\varphi\rangle$ and $|\varphi\rangle - P|\varphi\rangle$ are orthogonal to each other.

Prop: Projection P: orthogonal $\Leftrightarrow$ hermitian.

eg: $P_i : H \to H(i)$. $\quad |\varphi\rangle \mapsto |\varphi(i)\rangle$. $\left(|\varphi\rangle = \sum_{i=0}^{l-1} |\varphi(i)\rangle\right)$.

9°. Important decompositions.

①. Schur decomposition.

$A \in \mathbb{C}^{k,k}$. eigenvalues $\lambda_0 \cdots, \lambda_{l-1}$
algebraic multiplicities $m_0 \cdots m_{l-1}$.

then. $\exists$ $U \in \mathbb{C}^{k,k}$ — unitary.
$T \in \mathbb{C}^{k,k}$ — upper triangular matrix

$$T = \begin{pmatrix} \lambda_0 \cdots \lambda_0 & & & * \\ & \lambda_1 \cdots \lambda_1 & & \\ 0 & & \ddots & \\ & & & \lambda_{l-1} \cdots \lambda_{l-1} \end{pmatrix}$$

s.t. $A = UTU^*$.

②. The spectral theorem. → postulates of quantum mechanics

Def: $A \in \mathrm{End}(H)$ normal $\Leftrightarrow$ $A^*A = AA^*$.

Thm: $A \in \mathbb{C}^{k,k}$. normal.

$\lambda_0 \cdots \lambda_{l-1}$ eigenvalues

$m_0 \cdots m_{l-1}$ algebraic multiplicities.

$\Rightarrow$ $\exists$ $U \in \mathbb{C}^{k,k}$. unitary. s.t.

$$U^*AU = \begin{pmatrix} \lambda_0 \cdots \lambda_0 & & & \\ & \lambda_1 \cdots \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{l-1} \cdots \lambda_{l-1} \end{pmatrix}$$

$$U = (\underbrace{\vec{u}_{M_0} \cdots \vec{u}_{M_1-1}}\ \underbrace{\vec{u}_{M_1} \cdots \vec{u}_{M_2-1}} \cdots \underbrace{\vec{u}_{m_{l-1}} \cdots \vec{u}_{M_l-1}})$$

orthonormal basis
of eigenspace with (λ_i).

☆ Thm (Spectral theorem).

☆ Singular value decomposition.

$A \in \mathbb{C}^{(k,l)}$. $\mathrm{rank}(A) = r$.

then. $\exists$ $U \in \mathbb{C}^{k,k}$, $V \in \mathbb{C}^{l,l}$ unitary matrices.

s.t.

$$A = U \left(\begin{array}{c|c} \begin{matrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_{r-1} \end{matrix} & 0 \\ \hline 0 & 0 \end{array} \right) V^*.$$

where $\underbrace{\lambda_0 \cdots \lambda_{r-1}}_{\text{unique}}$ are positive real numbers.

Tensor Products. (key role in quantum mechanics)

① mathematical description of composite quantum systems.

② distinction between seperable and entangled $\frac{\text{states}}{\text{qubits}}$.

③ construction of quantum gates and quantum circuits.

④ formulation of density matrices for mixed states.

Example 2.5.1.

$H_1 = \sum_{b\in\{0,1\}} \mathbb{C}|b\rangle$. $B_1 = (|0\rangle, |1\rangle)$.

$\Rightarrow H = H_1 \otimes H_1$, with basis $B = (|0\rangle\otimes|0\rangle, |0\rangle\otimes|1\rangle, |1\rangle\otimes|0\rangle, |1\rangle\otimes|1\rangle)$.

$= (|00\rangle, |01\rangle, |10\rangle, |11\rangle)$

$= (|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle)$

$\dim(H) = 4$.

More general. $H = H(0) \otimes H(1) \otimes \cdots \otimes H(m-1)$.

dim $\downarrow$ k_0, $\downarrow$ k_1, $\downarrow$ k_m

Basis B_0, B_1, B_m.

$\Rightarrow \dim(H) = k = \prod_{j=0}^{m-1} k_j$. $B = B_0 \otimes \cdots \otimes B_{m-1}$.

$\mathbb{Z}_{\vec{k}} = \prod_{j=0}^{m-1} \mathbb{Z}_{k_j}$ with $\vec{k} = (k_0, \dots, k_{m-1})$.

For $\vec{i} \in \mathbb{Z}_{\vec{k}}$. $\vec{i} = (i_0, \cdots, i_{m-1})$. with $i_j \in \mathbb{Z}_{k_j}$.

$|b_{\vec{i}}\rangle = \bigotimes_{j=0}^{m-1} |b_{i_j,j}\rangle = |b_{i_0,0}\rangle \cdots |b_{i_{m-1},m-1}\rangle$. $|b_{s,j}\rangle \in H_j$.

Caculate with Basis

Prop 2.5.3. $0 \le j < m$, $|\varphi_j\rangle \in H(j)$. with $|\varphi_j\rangle = \sum_{i=0}^{k_j-1} \alpha_{i,j} |b_{i,j}\rangle$ basis of $H(j)$.

Then. $$\bigotimes_{j=0}^{m-1} |\varphi_j\rangle = \sum_{\vec{i} \in Z_{\vec{k}}} \alpha_{\vec{i}} |b_{\vec{i}}\rangle \qquad \vec{i} = (i_0, \cdots, i_{k_j-1}) \in Z_{\vec{k}}.$$

and. $$\alpha_{\vec{i}} = \prod_{j=0}^{m-1} \alpha_{i_j, j}.$$

Proof: $$\underbrace{|b_{i_0,0}\rangle \cdots |b_{i_j,j}\rangle}_{\cong |b_{\vec{i}}\rangle} \qquad \alpha_{i_0,0} \cdots \alpha_{i_j,j} \triangleq \alpha_{\vec{i}}.$$

so. $$\bigotimes_{j=0}^{m-1} |\varphi_j\rangle = \sum_{\vec{i} \in Z_{\vec{k}}} \alpha_{\vec{i}} \cdot |b_{\vec{i}}\rangle.$$

2.5.2. Inner product.

$H = H(0) \otimes \cdots H(m-1)$. $\quad B = B_0 \otimes \cdots \otimes B_{m-1}$.

Prop 2.5.5. $$\langle \underbrace{\bigotimes_{j=0}^{m-1} |\varphi_j\rangle}_{\varphi} \mid \underbrace{\bigotimes_{j=0}^{m-1} |\psi_j\rangle}_{\psi} \rangle = \prod_{j=0}^{m-1} \langle \varphi_j | \psi_j \rangle.$$

Proof: ① let. $|\varphi_j\rangle = \sum_{i=0}^{k_j-1} \alpha_{ij} \underbrace{|b_{i,j}\rangle}_{\text{basis}}$. $\quad |\psi_j\rangle = \sum_{i=0}^{k_j-1} \beta_{i,j} |b_{i,j}\rangle$.

$$\Rightarrow |\varphi\rangle = \sum_{\vec{i} \in Z_{\vec{k}}} \alpha_{\vec{i}} |b_{\vec{i}}\rangle. \quad |\psi\rangle = \sum_{\vec{i} \in Z_{\vec{k}}} \beta_{\vec{i}} |b_{\vec{i}}\rangle.$$

$$\Rightarrow \langle \varphi | \psi \rangle = \langle \sum_{\vec{i} \in Z_{\vec{k}}} \alpha_{\vec{i}} |b_{\vec{i}}\rangle \mid \sum_{\vec{i} \in Z_{\vec{k}}} \beta_{\vec{i}} |b_{\vec{i}}\rangle \rangle.$$

$$= \sum_{\vec{i} \in Z_{\vec{k}}} \overline{\alpha_{\vec{i}}} \cdot \beta_{\vec{i}}.$$

② $$\prod_{j=0}^{m-1} \langle \varphi_j | \psi_j \rangle = \prod_{j=0}^{m-1} \langle \sum_{i=0}^{k_j-1} \alpha_{ij} |b_{ij}\rangle \mid \sum_{i=0}^{k_j-1} \beta_{ij} |b_{ij}\rangle \rangle$$

$$= \prod_{j=0}^{m-1} \sum_{i=0}^{k_j-1} \overline{\alpha_{ij}} \beta_{ij} = \sum_{i=0}^{k_j-1} \overline{\alpha_{\vec{i}}} \beta_{\vec{i}}.$$

$H = H_{n_0} \otimes \cdots \otimes H_{n_{m-1}}$. with $n = \sum_{j=0}^{m-1} n_j$.

$H \to H_n$ $\quad |\vec{b}_0\rangle \cdots |\vec{b}_{m-1}\rangle \mapsto |\vec{b}_0 \vec{b}_1 \cdots \vec{b}_{m-1}\rangle$. isometry.

eg: $|0\rangle|0\rangle = |00\rangle$.

2.5.4. Homomorphisms. (quantum gate. circuits).

$H'_{(j)}$ ~~the dual space of $H_{(j)}$.~~

Matrix ~~expre~~ representation.

$\mathrm{Hom}(\underset{B}{H}, \underset{C}{H'}) \to \mathbb{C}^{(l,k)}$. $\quad f \mapsto \mathrm{Mat}_{B,C}(f)$.

$f_0 \in H_0$.

$$\begin{pmatrix} a_{11}\vec{\alpha} & a_{12}\vec{\alpha} \\ a_{21}\vec{\alpha} & a_{22}\vec{\alpha} \end{pmatrix} \otimes \left(\right)$$

f_0. $\qquad f_1$.

$A \in \mathbb{C}^{k_0, k_0'}$ $\qquad B \in \mathbb{C}^{k_1, k_1'}$.

B $\qquad$ B'.

$f \in \mathrm{Hom}(H \quad H')$.
$\quad\;\; \| \qquad\quad \|$
$H_0 \otimes H_1$. $\; H_0' \otimes H_1'$

A.

$f: \otimes |\varphi_0\rangle \otimes |\varphi_1\rangle \mapsto f_0(|\varphi_0\rangle) \otimes f_1(|\varphi_1\rangle)$.

$A|\varphi_0\rangle \otimes B|\varphi_1\rangle$.

$(A)(\alpha) \mapsto \otimes (B)(\beta)$.

$\underline{\varphi \otimes \psi}.$

$f = (f_0 \ f_1).$

$f_0 \in \mathrm{Hom}(H(0), H(0)')$

$A_0 \in \mathbb{C}^{k_0 k_0'}$

$A_1 \in \mathbb{C}^{k_1 k_1'}$

$\varphi \in H(0). \quad A_0\varphi \in \mathbb{C}^{k_0'}$

$\psi \in H(1). \quad A_1\psi \in \mathbb{C}^{k_1'}.$

$$A_0 \otimes A_1 = \begin{pmatrix} a_{11}A_1 & a_{12}A_1 \\ a_{21}A_1 & a_{22}A_1 \end{pmatrix} \begin{pmatrix} \varphi \\ \vdots \\ \psi \end{pmatrix} \begin{pmatrix} \psi \\ \vdots \\ b_2\psi \end{pmatrix}.$$

$$= \begin{pmatrix} a_{11}b_1A_1\psi + a_{12}b_1A_1\psi \\ a_{21}A_1\psi + a_{22}b_2A_1\psi \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_1 + a_{12}b_1 \\ \end{pmatrix} \otimes A_1\psi.$$

$$= A_0\varphi \otimes A_1\psi.$$

$$H(0) \otimes H(1). \longrightarrow H'(0) \otimes H'(1).$$

$$|\varphi\rangle \otimes |\psi\rangle \longrightarrow |A_0\varphi\rangle \otimes |A_1\psi\rangle.$$

$$B(|\varphi\rangle \otimes |\psi\rangle) = \_\_ \otimes \_\_.$$

$A_0 \otimes A_1$

Hadamard operator. $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

$(|0\rangle, |1\rangle)$

$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (|0\rangle, |1\rangle)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$.

$$(H \otimes H)(|0\rangle \otimes |0\rangle) = H|0\rangle \otimes H|0\rangle$$
$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$
$$= \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle.$$

$|\varphi_0\rangle$ ..

calculate eigenvalue.

$A_0|\varphi_0\rangle = \lambda_0|\varphi_0\rangle$ $\qquad A_{m-1}|\varphi_{m-1}\rangle = \lambda_{m-1}|\varphi_{m-1}\rangle$.

$$A\left(\bigotimes_{j=0}^{m-1}|\varphi_j\rangle\right) = A_0|\varphi_0\rangle \otimes \cdots \otimes A_{m-1}|\varphi_{m-1}\rangle$$
$$= \lambda_0 \cdots \lambda_{m-1}\,|\varphi_0\rangle \otimes \cdots |\varphi_{m-1}\rangle.$$

Example 2.5.17.

$H = H_1 \otimes H_1$. $\qquad A_0, A_1$ — Pauli X operator

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$|x_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. $\quad |x_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

$X(|x_+\rangle) = 1\cdot|x_+\rangle \quad X|x_-\rangle = (-1)|x_-\rangle$.

projection. $\quad |x_+\rangle\langle x_+| \quad |x_-\rangle\langle x_-|$.

$A = A_0 \otimes A_1$.

the eigenvalues of A are 1 and -1.

$$A|x_+x_+\rangle = A_0|x_+\rangle \otimes A_1|x_+\rangle = |x_+\rangle \otimes |x_+\rangle = |x_+x_+\rangle.$$

$$A|x_-x_-\rangle = A_0|x_-\rangle \otimes A_1|x_-\rangle = (-1)|x_-\rangle \otimes (-1)|x_-\rangle = |x_-x_-\rangle.$$

$$A|x_+x_-\rangle = -|x_+x_-\rangle. \qquad A|x_-x_+\rangle = -|x_-x_+\rangle.$$

eigenspaces.

E_1 $\quad B_1 = (|x_+x_+\rangle, |x_-x_-\rangle)$

E_{-1} $\quad B_{-1} = (|x_+x_-\rangle, |x_-x_+\rangle)$.

projection.

$$P_1 = |x_+x_+\rangle\langle x_+x_+| + |x_-x_-\rangle\langle x_-x_-|.$$

$$P_2 = |x_+x_-\rangle\langle x_-x_+| + |x_-x_+\rangle\langle x_-x_+|:$$

Thm. 2.5.18. (Schmidt decomposition theorem).

$|\varphi\rangle \in H(0) \otimes H(1)$. $\quad B_0 = (|u_0\rangle \cdots |u_{m-1}\rangle)$.

$\quad B_1 = (|v_0\rangle \cdots |v_{m-1}\rangle)$.

then. $|\varphi\rangle$ can be written (uniquely).

$$|\varphi\rangle = \sum_{i=0}^{m-1} r_i |u_i\rangle \otimes |v_i\rangle. \quad \text{with } r_i \text{ real numbers positive.}$$

there are.

Example: $|\varphi\rangle = |0\rangle \frac{|0\rangle + |1\rangle}{\sqrt{2}} \in H_1 \otimes H_1$. seperable.

$|\varphi\rangle = |\psi\rangle \otimes |\xi\rangle$
$\quad \in H(0) \quad \in H(1)$

B.A.C.

$|\varphi\rangle = BAC.$

$A = U \begin{pmatrix} D \\ 0 \end{pmatrix} V^*.$ (singular-value decomposition)

$U \in \mathbb{C}^{k,k}$ Unitary, $V \in \mathbb{C}^{l,l}$ Unitary. D: positive semidefinite diagonal matrix

$D = (r_0, \dots, r_{l-1}).$ $r_i \geq 0 . \in \mathbb{R}.$

$U = (U_1 \, U_2)$, $U_1 \in \mathbb{C}^{k,l}$, $U_2 \in \mathbb{C}^{k,k-l}.$ $\Rightarrow A = (U_1 \; \cancel{U_2}) \begin{pmatrix} D \\ 0 \end{pmatrix} V^*$

$= U_1 D V^*.$

$\Rightarrow |\varphi\rangle = \underbrace{B U_1}_{} D V^* C.$

check

$B U_1 = (|b_0\rangle \cancel{\dots |b_k\rangle}) \left(\quad \right)_{k \times l} = (|u_0\rangle, \dots, |u_{l-1}\rangle)$

$V^* C = \left(\quad \right)_{l \times l} (|c_0\rangle, \dots |c_{l-1}\rangle)^T = (|v_0\rangle, \dots |v_{l-1}\rangle)^T.$

$\Rightarrow |\varphi\rangle = \sum_{i=0}^{l-1} r_i \, |u_i\rangle \otimes |v_i\rangle.$