CHAPTER2: Complex numbers quaternions and geometry ①

Two number systems.

complex numbers $\mathbb{C} = \{a+bi \,;\, a, b \in \mathbb{R}\}$

quaternion $\mathbb{H}$ (Hamilton) $= \{a+bi+cj+dk : a.b.c.d \in \mathbb{R}\}$,

Note that, they all have multiplicative identity I. and.

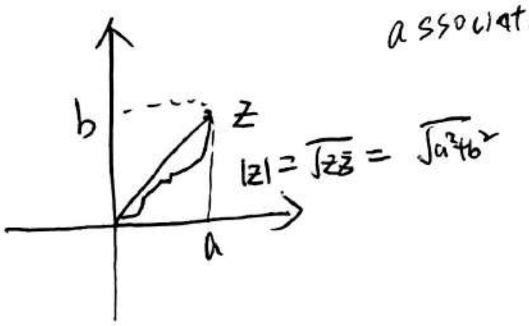
$|z|^2 = \bar{z} z$, $|z_1 z_2| = |z_1||z_2|$.

Do we have other number system ??? like eight dimensional, or sixteen dimensional?

Let start this chapter by putting these in our mind.

preparation: $\mathbb{C} = \{a+bi \,;\, a, b \in \mathbb{R}\}$. $z = a+bi$ $\bar{z} = a-bi$

associative, commutative)



If $|z|=1 \Rightarrow z = e^{i\alpha} = \cos\alpha + i\sin\alpha$.

like Kuramoto model. $z = \frac{1}{N}\sum_{i=1}^{N} e^{i\theta_i} = re^{iz}$ (r: order parameter.)

We can extend complex conjugation to matrices and vectors

$A = (a_{ij})$ $(a_{ij} \in \mathbb{C})$ $\bar{v} = (\bar{v}_1, \bar{v}_2, \cdots \bar{v}_n)$ $(\bar{v}_i \in \mathbb{C})$

Two important group in Chapter 6:

(1) Unitary group $U(n; \mathbb{C}) = \{M \in M(n, \mathbb{C}) ; \bar{M}^T M = I_n\}$

$M \in U(n; \mathbb{C}) \Rightarrow \det(M) = \pm 1$.

(2) Special unitary group $SU(n, \mathbb{C}) = \{M \in U(n, \mathbb{C}) \,;\, \det(M) = 1\}$

. quaternions. $\mathbb{H} = \{a+bi+cj+dk ,\ a.b.c.d \in \mathbb{R}\}$

(why??) $i^2 = j^2 = k^2 = -1$ $ij = k = -ji$ $jk = i = -kj$ $ki = j = -ik$.

Since Hamilton want to extend complex number to quaternions.

$i^2 = j^2 = k^2 = ijk = -1 \Rightarrow ij = k = -ji$. (associative, non-commutative addition: $Z = a+bi+cj+dk$ $\quad z' = a'+b'i+c'j+d'k$ $\quad z+z' = a+a' + (b+b')i + (c+c')j + (d+d')k$

multiplication

$$zz' = (aa'-bb'-cc'-dd') + (ab' + ba' - dc' + cd')i$$

$$+ (ac' + ca' + da' - ad')j + (ad' + da' + bc - cb')k.$$

What is identity for addition and multiplication??? 0. (1,0,0,0)

$(xz)y = (x(zy))$ $\qquad x(z+y) = xz + xy$.

associative $\qquad$ distributive law

$\bar{z} = a - bi - cj - dk$ $\qquad |z| = \sqrt{\bar{z}z} = \sqrt{z\bar{z}}$???

$= a^2+b^2+c^2+d^2$.

Main Results:

T2.1. Assume T is in $O(\mathbb{R}^3)$ and T has matrix M (standard basis) Then T has determinant $1 \Leftrightarrow T$ is a rotation about the line through the origin.

If line is the z-axis, we have

$$M = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

proof: $O(\mathbb{R}^3) = \{T \in GL(\mathbb{R}^3) : T$ preserves distance $\}$.

$GL(\mathbb{R}^3) = \{T : \mathbb{R}^n \to \mathbb{R}^n : T$ linear and invertible $\}$

We prove it into two steps.

Step A: T is a rotation $\Rightarrow \det(M) > 0$.

$T \in O(\mathbb{R}^3) = \{M \in M(3;\mathbb{R}),\ M^T M = I_3\} \Rightarrow \det(MM^T) = \det(M)\det(M^T) = (\det(M))^2 = \det(I_3) = 1$

$\Rightarrow \det(M) = \pm 1$ by T is a rotation $\Rightarrow \det(T) = 1$.

$T = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$	$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Step B: $\det(M)=1 \Rightarrow T$ is a rotation. ③

$\det(M)=1 \Rightarrow \det(I_3 - M) = \det(M^T M - M) = \det((M^T - I_3)M)$

$= \det(M^T - I_3)\underbrace{\det(M)}_{=1} = \det(M^T - I_3) = \det(M - I_3) = (-1)^3 \det(I_3 - M)$

$\Rightarrow \det(I_3 - M) = 0 \Rightarrow \exists a \neq 0$ st $(M - I_3)a = 0$.

$\Rightarrow a$ is the eigenvector of eigenvalue 1.

Choose space $U = \{u \in \mathbb{R}^3 : u \cdot a = 0\}$ $\dim(U) = 2$. a is the axis.

Next, we show that T is the rotation with axis a.

Step B.2 By $T \subset O(\mathbb{R}^3) \Rightarrow \|T(a) - T(b)\| = \|a - b\|$.

T preserve distance $\Rightarrow T$ is rotation.

$\forall x \in \mathbb{R}^3$ $x = \alpha a + \beta e_1 + \beta e_2$. ($e_1, e_2$ is the basis of U)

$T(x) = \alpha a + \beta T(e_1) + \beta T(e_2)$.

$T \in O(\mathbb{R}^3) \Rightarrow T$ preserve linear produce.
$T(a) = a$

Step B.1. $T(U) = U$.

Case B.1.1 $T(U) \subset U$. $\forall x \in T(U)$. $x \cdot a = T(v) \cdot T(a) = u \cdot a = 0$

$\Rightarrow x \cdot a = 0 \Rightarrow x \in U$.

Case B.1.2. $U \subset T(U)$. We only need to show $\dim(T(U)) = 2$.

$T(e_1) \cdot T(e_2) = e_1 \cdot e_2 = 0$ $T(\alpha e_1 + \beta e_2) = \alpha T(e_1) + \beta T(e_2)$

$\forall y \in T(U)$, $\exists x \in U = \alpha e_1 + \beta e_2$ $y = \alpha T(e_1) + \beta T(e_2)$.

The proof is completed.

If we take $a = (0,0,1)$ $U = xy$ axis.

$T: U \to U \Leftrightarrow T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear preserving distance.

by page 23 $\Rightarrow T' = \left\{ \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} \alpha \in \mathbb{R} \right\}$ ④

by $\det(M) = 1$ $\Rightarrow T' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$\Rightarrow M = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□. easy

T2.2. ⓐ $z \in \mathbb{C}$ $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$. $|z_1 z_2| = |z_1||z_2|$ ✓

ⓑ $z_1 z_2 = z_2 z_1$ (Abelian)

group: ① closure ② associative $(a+b)+c = a+(b+c)$

③ Identity ④ Ineverse element.

In summary: $(\mathbb{C}, +)$ is an abelian group.

ⓒ $(\mathbb{C} \setminus \{0\}, \times)$ is an abelian group.

$(a+bi)(a'+b'i) = (aa'-bb') + (ab'+ba')i$.

① closure ② associative ③ Identity (1.0).

④ Inverse element. $(a+bi)(a'+b'i) = (1.0) \Rightarrow a' = \frac{a}{a^2+b^2}$ $b' = -\frac{b}{a^2+b^2}$

$\begin{cases} ab' + ba' = 0 \\ aa' - bb' = 1. \end{cases}$ $\Leftarrow a' = \frac{1+bb'}{a}$ $ab' + \frac{b}{a}(1+bb') = 0 \Rightarrow b'\left(\frac{b^2}{a} + a\right) = -\frac{b}{a}$

$\Rightarrow b' = \frac{-\frac{b}{a}}{\frac{a^2+b^2}{a}} = -\frac{b}{a^2+b^2}$ $a' = \frac{a}{a^2+b^2}$

$(S^1, \times)$ is a subgroup of $(\mathbb{C}, \times)$.

① S^1 is closed ② S contains $(1,0)$ ③ Inverse $a^2+b^2=1 \Rightarrow a'=a$ $b'=-b$.

since $|z_1||z_2| = |z_1 z_2| = 1$

□.

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T2.3. $U(n;\mathbb{C})$ is a group under matrix multiplication, and $SU(n;\mathbb{C})$ is a subgroup. In the special case $n=2$. every matrix in $SU(2;\mathbb{C})$ has the form

$$M=\begin{bmatrix} u & -\bar{v} \\ v & \bar{u} \end{bmatrix} \text{ for } u, v \text{ satisfying } u\bar{u}+v\bar{v}=1$$

proof: (*) $U=\{M(n;\mathbb{C}): \bar{M}M^T=I\}$ is a group under ×

① closed. $\bar{M}_1^T M_1=I$ $\bar{M}_2 M_2=I$ $(\overline{M_1M_2})^T M_1M_2=\bar{M}_2^T\bar{M}_1^T M_1M_2=I$.

② associative $M_1(M_2M_3)=(M_1M_2)M_3$.

③ identity $I_n=\begin{bmatrix} (1,0) & & \\ & (1,0) & \\ & & \ddots (1,0) \end{bmatrix}$

④ Inverse $\bar{M}^T M=I_n \Rightarrow \bar{M}^T=M^{-1}$ → $M\bar{M}^T=I_n$. why ???

$\bar{M}^T=\bar{M}^T MC = IC = C. \Rightarrow \bar{M}^T=C \Rightarrow M^{-1}=M^T$

(*) $SU(n;\mathbb{C})=\{M\in U(n,\mathbb{C})\ ;\ \det(M)=1\}$.

① closed $\det(M_1M_2)=\det M_1 \cdot \det(M_2)=1$.

② associative ✓

③ identity $I_n\in SU(n,\mathbb{C})$ $\qquad \det(\bar{M}^T)=\overline{\det(M)}=1$.

④ Inverse $\bar{M}^T\in SU(n;\mathbb{C})$ If $M\in SU(n;\mathbb{C})$

(*) $n=2. \Rightarrow M=\begin{bmatrix} u & -\bar{v} \\ v & \bar{u} \end{bmatrix}$

We set $M=\begin{bmatrix} u & w \\ v & z \end{bmatrix}$ $\det(M)=1 \Rightarrow uz-vw=1$ (☆)

$$\bar{M}^TM=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} \bar{u} & \bar{w} \\ \bar{v} & \bar{z} \end{bmatrix}^T\begin{bmatrix} u & w \\ v & z \end{bmatrix}=\begin{bmatrix} \bar{u} & \bar{v} \\ \bar{w} & \bar{z} \end{bmatrix}\begin{bmatrix} u & w \\ v & z \end{bmatrix}=\begin{bmatrix} \bar{u}u+\bar{v}v & \bar{u}w+\bar{v}z \\ \bar{w}u+\bar{z}v & \bar{w}w+\bar{z}z \end{bmatrix}$$

$\Rightarrow \bar{u}u+\bar{v}v=1$ $\bar{w}w+\bar{z}z=1$ → $v\bar{v}=1 \Rightarrow \frac{1}{v}=\bar{v}$

Case A. $u=0$ by ☆ $\Rightarrow vw=-1 \Rightarrow w=-v^{-1}=-\bar{v}$ why ???

Since $\begin{matrix}\bar{u}w+\bar{v}z=0 \\ \bar{w}u+\bar{z}v=0\end{matrix}$ + (☆) $\Rightarrow w=-\frac{\bar{v}z}{\bar{u}}$ (☆☆) ⟸ Case B $u\neq 0$

$w = -\frac{\bar{v}z}{\bar{u}}$ $w = \frac{-1+uz}{v}$ $z = \bar{u}$ (why ???) $\Rightarrow w = -\bar{v}$

$\Rightarrow \bar{u} - \bar{u}uz = v\bar{v}z \Rightarrow (\bar{u}u + v\bar{v})z = \bar{u}$

T2.4. (A) Quaternionc conjugation.

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ $\overline{z_1 z_2} = \bar{z}_2 \bar{z}_1$

We cant set $z_1 = a_1 + w_1$ $z_2 = a_2 + w_2$

$w_1 = (i, j, k)$ $w_2 = (i, j, k)$.

$\bar{z}_1 = a_1 - w_1$ $\bar{z}_2 = a_2 - w_2$

$\Rightarrow \overline{z_1 z_2} = a_1 a_2 - a_1 w_2 - a_2 w_1 - w_1 w_2 = \bar{z}_2 \bar{z}_1$

$\overline{a_1 a_2 + a_1 w_1 + a_2 w_2 + w_1 w_2} = a_1 a_2 - w_1 w_2 - a_2 w_1 - a_1 w_2$

(B) The set $(\mathbb{H}, +)$ is an abelian group.

1. closed 2. identity 0. 3. associative
4. Inevrse $(-a, -b, -c, -d)$ 5. $z_1 + z_2 = z_2 + z_1$ (commutation)

(C) $(\mathbb{H} \setminus \{0\}, \times)$ is a non-abelian group.

1. Closed. $z\bar{z} = a^2 + b^2 + c^2 + d^2$.
2. Identity $1 = (1, 0, 0, 0)$ 3 associative. $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
4. Inevrse $z^{-1} = \frac{\bar{z}}{a^2 + b^2 + c^2 + d^2} = 1$
5. non-abelian $(0, 0, 1, 0)(0, 1, 0, 0) = (0, 0, 0, -1)$

$(0, 1, 0, 0)(0, 0, 1, 0) = (0, 0, 1, 0)$

(D) The 3-sphere S^3 is a non-abelian subgroup: of $(\mathbb{H} \setminus \{0\}, \times)$

1. Closed $|z_1|=1$ $|z_2|=1$ $|z_1 z_2| = |z_1||z_2| = 1$

$= \sqrt{z_1 z_2 \overline{z_1 z_2}} = \sqrt{z_1 z_2 \bar{z}_2 \bar{z}_1} = 1$

2. identity $(1, 0, 0, 0) \in S^3$

3. $|z| = |\bar{z}| = 1$ $z^{-1} = \bar{z}$ (Inverse)

4. $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ associative.

T2.5. (A). Every quaternion q in S^3 can be written as

(why ???) $q = \cos(\frac{\alpha}{2}) + \sin(\frac{\alpha}{2}) u$. $u \in S^3$ (Real(u)=0)

$a^2+b^2+c^2+d^2=1 \Rightarrow |q| = \cos(\frac{\alpha}{2})^2 + \sin(\frac{\alpha}{2})^2 = 1$.

$\bar{q} = \cos\frac{\alpha}{2} - \sin\frac{\alpha}{2} u$. $q^{-1} = \bar{q}$

(B). For $q \in S^3$, $v \in \mathbb{R}^3$. $T_q(v) = q v q^{-1}$ defines a rotation. in $SO(\mathbb{R}^3)$. If $q = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} u \Rightarrow T_q$ is a rotation through angle α around an axis spanned by u.

1. well-defined $T: \mathbb{R}^3 \to \mathbb{R}^3$ $\overline{q v q^{-1}} = \bar{q}^{-1} \bar{v} \bar{q} = q(-v) q^{-1} = -q v q^{-1}$

$\Rightarrow Re(T_q(v)) = 0 \Rightarrow T_q(v) \in \mathbb{R}^3$.

2. $|T_q(v)| = |v|$ since $|q v q^{-1}| = |q||v||q^{-1}| = |v| \Rightarrow T_q \in O^3$.

$T_1 = V \Rightarrow \det(T_1) = 1$ $\det(T_q) = \det(T_1) = 1$ ⇐ T_q is continuous, det is continuous

3. T is linear $T_q(v+w) = T_q(v) + T_q(w)$ $T_q(rv) = r T_q(v)$

4. $\det(T) = 1 \Rightarrow T$ is a rotation.

Actually if $q = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} u$. T_q $|u, v, w|$ ($U = \{x \cdot u = 0\}$ v, w is basis of U)

$$T(u, v, w) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Denote $q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} u = c + su$ $\quad u \in S^3 \quad u^2 = -1$.

$q^{-1} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2} u =: c - su$.

For any $x \in \mathbb{R}^3$.

$T_q(x) = qxq^{-1}$ (Rodrigues formula) (☆)

Note that $\vec{u}\vec{v} = -(\vec{u}\cdot\vec{v}) + \vec{u}\times\vec{v}$

$uv = (u_1, u_2, u_3)(v_1, v_2, v_3) = (-u_1v_1 - u_2v_2 - u_3v_3) + (u_2v_3 - u_3v_2)i + (u_3v_1 - u_1v_3)j + (u_1v_2 - u_2v_1)k = -(u\cdot v) + \vec{u}\times\vec{v}$

By (☆) $T_q(x) = cxc - cxsu + suxc - s^2uxu$.

$= c^2x - cs\,xu + sc\,ux - s^2uxu$.

$=: c^2x + I_1 + I_2 + I_3$

$I_1 = -cs\,xu = cs(x\cdot u - \vec{x}\times\vec{u})$. $\quad \because a\times b = (b\times a)\cdot(-1)$

$I_2 = sc\,ux = cs(-u\cdot x + \vec{u}\times\vec{x})$ $\quad \Rightarrow I_1 + I_2 = -2cs(x\times u)$

$I_3 = -s^2uxu = -s^2(-u\cdot x + \vec{u}\times\vec{x})\vec{u}$.

$= s^2(u\cdot x)\vec{u} - s^2((\vec{u}\times\vec{x})\vec{u})$

$= s^2(u\cdot x)\vec{u} - s^2(\underbrace{-(\vec{u}\times\vec{x})\cdot\vec{u}}_{=0} + \vec{u}\times\vec{x}\times\vec{u})$

$= s^2(u\cdot x)\vec{u} - s^2(x(u\cdot u) - u(x\cdot u))$

$\overset{u\cdot u=1}{=} -s^2x + s^2u(x\cdot u)$.

$\Rightarrow T_q(x) =: (c^2 - s^2)x + 2cs(u\times x) + s^2u(x\cdot u)$

$c^2 - s^2 = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta \quad 2cs = \sin\theta \quad 2s^2 = 1 - \cos\theta$

$\Rightarrow T_q(x) = x\cos\theta + (\vec{u}\times\vec{x})\sin\theta + u(u\cdot x)(1 - \cos\theta)$.

$T_q(u) = u \Rightarrow$ If $u = v\times w$, w, v are the basis.

$T_q(v) = v\cos\theta + (u \times v)\sin\theta = v\cos\theta - w\sin\theta$ (9)

$T_q(w) = w\cos\theta + (u \times w)\sin\theta = v\sin\theta + w\cos\theta$

$$\Rightarrow M = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow T \text{ is a rotation by T2.1}$$

(c) $F: S^3 \to SO(\mathbb{R}^3)$ is onto but not one-to-one.

$(u \Rightarrow -u,\ \alpha \to \alpha + 2\pi)$

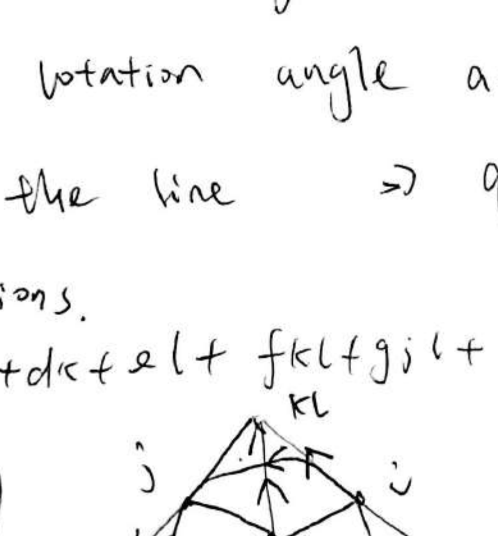
proof: $F(q) = q v q^{-1}$ $\forall\, v \in SO(\mathbb{R}^3)$ $\exists\, q$ s.t. $F(q) = T_q = v$

by T2.1 $T \in SO(\mathbb{R}^3) \Rightarrow T$ is a rotation about the line through the origin. Take α be the rotation angle and u be the unit vector for the line $\Rightarrow q = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} u$ (onto) ☐

2.5 Octonions.

$\mathbb{O} = \{a + bi + cj + dk + el + fkl + gjl + hil,\ a,b,c,d,e,f,g,h \in \mathbb{R}\}$

String theory



$(kl)j = il$

$j(kl) = -il$

Question: Can you create higher and higher dimensional number system??: $\mathbb{R}\ \mathbb{C}\ \mathbb{H}\ \mathbb{O}$

If we require identity to be I $|z|^2 = z\bar{z}$ $|z_1 z_2| = |z_1||z_2|$

Then ~~No~~ !!! by Hurwitz.