Lie Groups by Harriet Pollatsek (Chapter 1)

1. Preliminaries

Def: • $(G,*)$ **group** if

- $g,h \in G \Rightarrow g*h \in G$
- $\exists e \in G : \ g*e = e*g = g \quad \forall g \in G$
- $\forall g \in G \ \exists g^{-1} \in G : \quad g*g^{-1} = g^{-1}*g = e$
- $g*(h*k) = (h*k)*g \quad \forall g,h,k \in G$

• Suppose $(G,*)$ is a group.
$H \subseteq G$ is a subgroup of G if $(H,*)$ is a group.

Def: $T: X \to Y$ **invertible** if

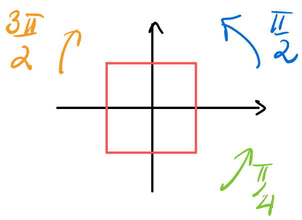
$$\exists S: Y \to X : \quad S(T(P)) = P \quad \forall P \in X$$
$$T(S(Q)) = Q \quad \forall Q \in Y$$

$T^{-1} := S$ **inverse of T**

2. Symmetries

: invertible functions from some set to itself preserving some feature of the set.

e.g. 1. **Rotations**



$\circlearrowleft \frac{\pi}{2}$ preserves the square

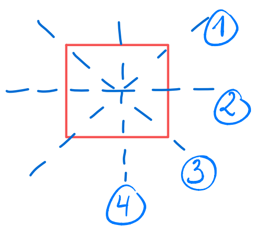
$\frac{3\pi}{2} \circlearrowleft$ is the same as $\circlearrowleft \frac{\pi}{2}$

$\circlearrowleft \frac{\pi}{4}$ doesn't preserve the square

If $T := \circlearrowleft \alpha$, then the inverse of T is $T^{-1} = \circlearrowright \alpha$

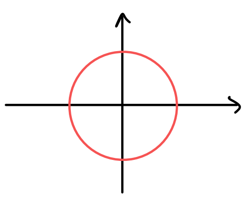
The square $\square$ has 4 rotation symmetries: $0\circlearrowleft, \frac{\pi}{2}\circlearrowleft, \pi\circlearrowleft, \frac{3\pi}{2}\circlearrowleft$

2. **Reflections**



The square has 4 reflection symmetries

Another example:



The circle has "continuous" symmetry

3. Groups of symmetries

① Group of symmetries of $\mathbb{R}^n$

<u>Def:</u> $T: \mathbb{R}^n \to \mathbb{R}^n$ vector space symmetry of $\mathbb{R}^n$ if

- $\forall v, w \in \mathbb{R}^n$ $\quad T(v+w) = T(v) + T(w)$
  $\quad T(av) = aT(v)$
- T invertible

So T is an invertible linear transformation

<u>Recall:</u> • $T: \mathbb{R}^n \to \mathbb{R}^n$ linear transformation

$$\Rightarrow \quad \exists M_T \in \mathcal{M}(n, \mathbb{R}) \; : \; T(v) = M_T v \quad \forall v \in \mathbb{R}^n$$

• Conversely, every $M \in \mathcal{M}(n,\mathbb{R})$ defines a lin. transformation

$$T_M : \mathbb{R}^n \to \mathbb{R}^n, \quad v \mapsto Mv.$$

General linear group:
$$GL(n, \mathbb{R}) = \{ M \in \mathcal{M}(n, \mathbb{R}) : \det M \neq 0 \}$$
$$\cong GL(\mathbb{R}^n) = \{ T : \mathbb{R}^n \to \mathbb{R}^n \; ; \; T \text{ linear and invertible} \}$$

→ <u>Claim:</u> $(GL(n, \mathbb{R}), \cdot_{\mathcal{M}(n,\mathbb{R})})$ and $(GL(\mathbb{R}^n), \circ)$ are groups

<u>Pf:</u> Ex (use $\det(MN) = \det M \cdot \det N$)

Special linear group:
$$SL(n, \mathbb{R}) = \{ M \in \mathcal{M}(n, \mathbb{R}) : \det M = 1 \}$$
$$\cong SL(\mathbb{R}^n) = \{ T : \mathbb{R}^n \to \mathbb{R}^n : \det M_T = 1 \}$$

→ <u>Claim:</u> – $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$

– $SL(\mathbb{R}^n)$ is a subgroup of $GL(\mathbb{R}^n)$.

② Group of symmetries of Euclidean space $\mathbb{R}^2$

→ preserve Euclidean distance

$T: \mathbb{R}^2 \to \mathbb{R}^2$: – T invertible
– T linear transformation $\qquad (*)$
– T isometry: $\forall P, Q \in \mathbb{R}^2 \quad \| T(P) - T(Q) \|_2 = \| P - Q \|_2$

Orthogonal group in Euclidean sense: $O(\mathbb{R}^2) := \{ T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ as in } (*) \}$

→ <u>Claim:</u> $(O(\mathbb{R}^2), \circ)$ is a group

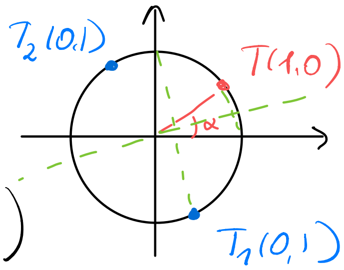
<u>Pf:</u> Ex

reflection across a line through $(0,0)$ w/ angle $\alpha/2$ with positive x-axis

→ <u>Claim:</u>
$$O(2, \mathbb{R}) = \left\{ \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}}_{R_\alpha}, \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} : \alpha \in [0, 2\pi) \right\}$$

Pf: $\subseteq$: Let $M \in O(2,\mathbb{R}) := \{ M \in \mathcal{M}(2,\mathbb{R}) \ : \ T_M \in O(\mathbb{R}^2) \}$. Then

- $\| T_M(1,0) \|_2 = \| (1,0) - (0,0) \|_2 = 1$, $\| T(0,1) \|_2 = 1$
- $T(1,0) = (\cos\alpha, \sin\alpha)$, $\alpha \in [0, \pi/2]$
- $\| T(1,0) - T(0,1) \|_2 = \sqrt{2}$



$\Rightarrow T_1(0,1) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$ or $T_2(0,1) = \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix}$

$\supseteq$: easy $\square$

$\rightarrow$ Claim: $O(2,\mathbb{R})$ is a group

Pf: Ex:
$$T_\alpha^{rot} \circ T_\beta^{rot} = T_{\alpha+\beta}^{rot} = T_\beta^{rot} \circ T_\alpha^{rot}$$
$$T_\beta^{rot} \circ T_\alpha^{ref} = T_{\alpha+\beta/2}^{ref}$$
$$T_\alpha^{ref} \circ T_\beta^{rot} = T_{\alpha-\beta/2}^{ref}$$
$$T_\alpha^{ref} \circ T_\beta^{ref} = T_{\alpha-\beta}^{rot}$$

Remark:
- set of all rotations is a group:

  $SO(2,\mathbb{R})$ special orthogonal group

- set of reflections is NOT a group

③ Group of symmetries of Euclidean space $\mathbb{R}^3$ ($\mathbb{R}^n$)

Preserving the distance = Preserving the dot product

$$\|p\|_2 = \sqrt{p \cdot p}$$

$$\|T(v) - T(w)\|_2 = \|v-w\|_2 \iff T(v) \cdot T(w) = v \cdot w \quad \forall v,w \in \mathbb{R}^n$$

Pf: $\Leftarrow$: $\|T(v) - T(w)\|_2 \overset{T \text{ lin.}}{=} \|T(v-w)\|_2 = \sqrt{T(v-w) \cdot T(v-w)}$
$$= \sqrt{(v-w)\cdot(v-w)} = \|v-w\|_2$$

$\Rightarrow$: $\|T(v) - T(w)\|_2 = \|v-w\|_2 \Rightarrow (T(v) - T(w)) \cdot (T(v) - T(w)) = (v-w)\cdot(v-w)$

$\Rightarrow T(v) \cdot T(v) - 2\,T(w) \cdot T(v) + T(w) \cdot T(w) = v \cdot v - 2\, v \cdot w + w \cdot w$

Take $w = 0$: $\|T(v) - T(0)\|_2 = \|T(v)\|_2 = \|v - 0\|_2 = \|v\|_2 \quad \forall v \in \mathbb{R}^n$

$\Rightarrow \|v\|_2^2 - 2\,T(w) \cdot T(v) + \|w\|_2^2 = \|v\|_2^2 - 2\, v \cdot w + \|w\|_2^2$

$\Rightarrow T(w) \cdot T(v) = w \cdot v \quad \forall w,v \in \mathbb{R}^n$

$\square$

$O(\mathbb{R}^3) = \{T \in GL(\mathbb{R}^3) : T \text{ preserves the dot product}\}$ (Euclidean) orthogonal group

→ Claim: $O(\mathbb{R}^3)$ is a group

Pf: - $I \in O(\mathbb{R}^3)$

- $S, T \in O(\mathbb{R}^3) \Rightarrow T \circ S \in O(\mathbb{R}^3)$, since

$$(T \circ S)(v) \cdot (T \circ S)(w) = T(S(v)) \cdot T(S(w)) \overset{T \in O(\mathbb{R}^3)}{=} S(v) \cdot S(w) \overset{S \in O(\mathbb{R}^3)}{=} v \cdot w$$

- $T \in O(\mathbb{R}^3) \Rightarrow T^{-1} \in O(\mathbb{R}^3)$, since

$\forall x, y \in \mathbb{R}^3 \ \exists v, w \in \mathbb{R}^3 : T(v) = x, \ T(w) = y$

$$\Rightarrow T^{-1}(x) \cdot T^{-1}(y) = v \cdot w = T(v) \cdot T(w) = x \cdot y$$

$\square$

$SO(\mathbb{R}^3) = \{T \in O(\mathbb{R}^3) : \det(T) = 1\}$ rotation group / special orthogonal group

→ subgroup of $O(\mathbb{R}^3)$

Q: What is the matrix group corresponding to $O(\mathbb{R}^3)$?

A: $O(3, \mathbb{R}) = \{M \in \mathcal{M}(3, \mathbb{R}) : M^T M = I_3\}$

→ Pf: Let $S : \mathbb{R}^3 \to \mathbb{R}^3 : S(v) = Mv \ \forall v \in \mathbb{R}^3$.

$$S(v) \cdot S(w) = (Mv) \cdot (Mw) = (Mv)^T (Mw) = v^T M^T M w = v^T (M^T M) w \quad (*)$$

Therefore $S \in O(\mathbb{R}^3) \iff M^T M = I_3$ (take $v = e_j$, $w = e_i$ in $(*)$)

$\square$

→ Claim: $O(3, \mathbb{R})$ is a group

Pf: - $I_3 \in O(3, \mathbb{R})$

- $M_1, M_2 \in O(3, \mathbb{R}) \Rightarrow (M_1 M_2)^T (M_1 M_2) = M_2^T (M_1^T M_1) M_2 = M_2^T M_2 = I_3$

$$\Rightarrow M_1 M_2 \in O(3, \mathbb{R})$$

- $M \in O(3, \mathbb{R}) \Rightarrow M^T M = I_3 \Rightarrow \det M^T \cdot \det M = 1 \Rightarrow (\det M)^2 = 1$

$\Rightarrow \det M = \pm 1 \neq 0$

$\Rightarrow \exists M^{-1} : M \cdot M^{-1} = I_3$

$\Rightarrow M^T M \cdot M^{-1} = I_3 \cdot M^{-1} \Rightarrow M^T = M^{-1}$

$$\Rightarrow (M^{-1})^T M^{-1} \overset{M^T = M^{-1}}{=} (M^T)^T M^{-1} = M \cdot M^{-1} = I_3$$

$\square$

Remark: $O(n, \mathbb{R}) = \{M \in \mathcal{M}(n, \mathbb{R}) : M^T M = I_n\}$ orthogonal group of degree n over real numbers

$SO(n, \mathbb{R}) = \{M \in O(n, \mathbb{R}) : \det M = 1\}$ special orthogonal group of degree n over real numbers

④ Group of symmetries of Lorentz plane $\mathbb{R}^2$ (spacetime)

Def. & Thm: Lorentz product on $\mathbb{R}^2$: $(t_1 x_1)\cdot(t_2 x_2) := t_1 t_2 - x_1 x_2$

has the following properties:
- symmetric
- bilinear
- nondegenerate

Pf: Ex

Lorentz group: $\mathcal{L}(\mathbb{R}^2) = \{T: \mathbb{R}^2 \to \mathbb{R}^2 : T \text{ linear}, \ T(v)\cdot T(w) = v\cdot w \ \ \forall v,w \in \mathbb{R}^2\}$

→ Claim: $(\mathcal{L}(\mathbb{R}^2), \circ)$ is a group

Pf:
- $I_2 \in \mathcal{L}(\mathbb{R}^2)$
- $S,T \in \mathcal{L}(\mathbb{R}^2) \Rightarrow T\circ S \in \mathcal{L}(\mathbb{R}^2)$
- $T\in \mathcal{L}(\mathbb{R}^2)$, $T(v) = 0 \Rightarrow v\cdot w = T(v)\cdot T(w) = 0 \quad \forall w \in \mathbb{R}^2$
  $\Rightarrow v = 0$

  $\Rightarrow T$ invertible

  Let $v,w \in \mathbb{R}^2$. Then $\exists\, x,y \in \mathbb{R}^2$ s.t. $T(x) = v$, $T(y) = w$. Then

  $T^{-1}(v)\cdot T^{-1}(w) = x\cdot y = T(x)\cdot T(y) = v\cdot w$

$\square$

Group of Lorentz rotations $\mathcal{L}^+(\mathbb{R}^2) = \{T \in \mathcal{L}(\mathbb{R}^2) : \det(T) = 1\}$

→ Claim: $\mathcal{L}^+(\mathbb{R}^2)$ is a subgroup of $\mathcal{L}(\mathbb{R}^2)$

Pf: Ex.

Q: What is the matrix group corresponding to $\mathcal{L}(\mathbb{R}^2)$?

A: $\mathcal{L}(2,\mathbb{R}) = \{M \in \mathcal{M}(2,\mathbb{R}) : M^T C M = C\}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\left[\ \mathcal{L}(4,\mathbb{R}) = \{M \in \mathcal{M}(4,\mathbb{R}) : M^T C M = C\}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\ \right]$$

→ Pf: 1) We show $\mathcal{L}(2,\mathbb{R})$ corresponds to $\mathcal{L}(\mathbb{R}^2)$.

Let $S: \mathbb{R}^2 \to \mathbb{R}^2$ st. $S(v) = Mv \quad \forall v \in \mathbb{R}$

$S(v)\cdot S(w) = (Mv)\cdot(Mw) = (Mv)^T C (Mw) = v^T (M^T C M) w$

Therefore $S \in \mathcal{L}(\mathbb{R}^2) \Longleftrightarrow M^T C M = C$

2) We know that $\mathcal{L}(2,\mathbb{R})$ is a group:

- $I \in \mathcal{L}(2,\mathbb{R})$

- $M_1, M_2 \in \mathcal{L}(2,\mathbb{R}) \Rightarrow (M_2M_1)^T C (M_2M_1) = M_1^T(M_2^T C M_2)M_1 \overset{M_2\in\mathcal{L}(2,\mathbb{R})}{=} M_1^T C M_1 \overset{M_1\in\mathcal{L}(2,\mathbb{R})}{=} C$

- Let $M \in \mathcal{L}(2,\mathbb{R})$. Then

$$M^TCM = C \Rightarrow (\det M)^2 \cdot \underbrace{\det C}_{=-1} = \det C$$

$$\Rightarrow \det M = \pm 1 \neq 0$$

$$\Rightarrow \exists M^{-1} : MM^{-1} = M^{-1}M = I$$

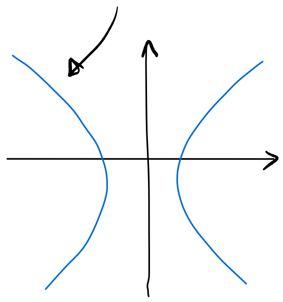
$$\Rightarrow (M^{-1})^T M^T C M M^{-1} = (M^{-1})^T C M^{-1}$$

$$\Rightarrow \underbrace{(MM^{-1})^T}_{=I} C \underbrace{MM^{-1}}_{=I} = (M^{-1})^T C M^{-1}$$

<u>Def</u>: $\sinh(\alpha) = \dfrac{e^{\alpha}-e^{-\alpha}}{2}$, $\cosh(\alpha) = \dfrac{e^{\alpha}+e^{-\alpha}}{2}$, hyperbolic trigonometric functions

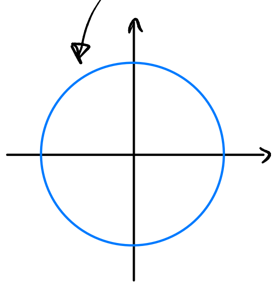
$\alpha \in \mathbb{R}$

→ $\forall (x,y) : x^2 - y^2 = 1 \Rightarrow \exists \alpha \in \mathbb{R} : (x,y) = (\cosh\alpha, \sinh\alpha)$



→ <u>Compare</u>: $\sin(\alpha)$, $\cos(\alpha)$, $\alpha \in \mathbb{R}$ circular trigonometric functions

$\forall (x,y) : x^2 + y^2 = 1 \Rightarrow \exists \alpha \in \mathbb{R} : (x,y) = (\cos\alpha, \sin\alpha)$



$\mathcal{L}^{++}(2,\mathbb{R})$

<u>Claim:</u> $\mathcal{L}(2,\mathbb{R}) = \left\{ \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}, \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ \sinh(\alpha) & -\cosh(\alpha) \end{pmatrix}, \begin{pmatrix} -\cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & -\cosh(\alpha) \end{pmatrix}, \begin{pmatrix} -\cosh(\alpha) & -\sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \right\}_{\alpha \in \mathbb{R}}$

rotation of 1st kind, reflection of 1st kind, rotation of 2nd kind, reflection of 2nd kind

$\mathcal{L}^{+}(2,\mathbb{R})$

<u>Pf:</u> Let $M \in \mathcal{L}(2,\mathbb{R})$ and let T be the corresponding linear transformation. Let $T(1,0) = (a,b)$, $T(0,1) = (c,d)$

Assume $a > 0$.

- $T(1,0) \cdot T(1,0) = (1,0)\cdot(1,0) = 1 \Rightarrow a^2 - b^2 = 1$

$\Rightarrow \exists\, \alpha \in \mathbb{R}: \quad T(1,0) = (\cosh(\alpha), \sinh(\alpha))$

- $T(1,0) \cdot T(0,1) = (1,0)\cdot(0,1) = 0 \Rightarrow ac - bd = 0$

$$\Rightarrow c = \frac{bd}{a} = \frac{\sinh(\alpha)}{\cosh(\alpha)}\, d$$

- $T(0,1) \cdot T(0,1) = (0,1)\cdot(0,1) = -1 \Rightarrow c^2 - d^2 = -1$

$$\Rightarrow \left[1 - \left(\frac{\sinh(\alpha)}{\cosh(\alpha)}\right)^2\right] d^2 = 1$$

$$\Rightarrow \frac{\cosh^2(\alpha) - \sinh^2(\alpha)}{\cosh^2(\alpha)}\, d^2 = 1$$

$$\Rightarrow d^2 = \cosh^2(\alpha) \Rightarrow d = \pm \cosh(\alpha)$$

$$\Rightarrow T(0,1) = \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \sinh(\alpha) \\ \cosh(\alpha) \end{pmatrix} \text{ or } T(0,1) = \begin{pmatrix} -\sinh(\alpha) \\ -\cosh(\alpha) \end{pmatrix}$$

Assume $a < 0$: Ex.

<u>Claim:</u> $\mathcal{L}^{++}(2,\mathbb{R})$ is a subgroup of $\mathcal{L}(2,\mathbb{R})$

$\mathcal{L}^{+}(2,\mathbb{R})$ is a subgroup of $\mathcal{L}(2,\mathbb{R})$

<u>Pf:</u> Ex.

(5) <u>Generalization:</u> $T: \mathbb{R}^n \to \mathbb{R}^n$ linear transformation

$v \cdot w$ nondegenerate symmetric bilinear form

$v \cdot w = v^T C w$

Then $v \cdot w = T(v) \cdot T(w) \iff M^T C M = C$

$\to O(\mathbb{R}^n)$ general orthogonal group

<u>Remark:</u> – If $C = M^T C M$ is anti-symmetric ($C^T = -C$), then $Sp(n, \mathbb{R})$ is the symplectic group (classical Lie group)

– By Sylevester's theorem, every finite-dim. real vector space with nondegenerate symmetric bilinear form has a basis s.t.

$C = \mathrm{diag}(\pm 1)$

$r := \#(+1)$ = signature of the form