Ch4. One parameter subgroups and the exponential map.

Definition. *(1)* ***(Matrix) Lie group*** *is a closed subgroup G of $GL(n,\mathbb{R})$. i.e., G is a subgroup of $GL(n,\mathbb{R})$ and satisfies the following condition: If a sequence of matrices $\{A_k\} \subset G$ converges in $GL(n,\mathbb{R})$, then $\lim_{k\to\infty} A_k \in G$. (Use $G < GL(n,\mathbb{R})$ to denote that G is a matrix Lie group.)*

Since $L(G)$ is defined as the collection of derivatives of smooth curves in G, one may ask how many different smooth curves we actually need to describe it. To answer this question, we will study one-parameter subgroups and the matrix exponential today.

In this lecture, I will assume that G is a Lie gp without futher mention.

Recall G Lie gp.

$$L(G) := T_I G = \{\gamma'(0) \mid \gamma : \mathbb{R} \to G \text{ smooth } \& \ \gamma(0) = I\}$$

How many curves we need? to specify $T_I G$

<Goal>
- learn one parameter subgp.
- learn matrix exponential.
- specify $L(G)$
- $\det(e^A) = e^{tr(A)}$

4.1 One-parameter subgps.

$\therefore L(G) = \{A \in M(n,\mathbb{R}) : e^{tA} \in G \text{ for } \forall t \in \mathbb{R}\}$

Def A map $\gamma : \underset{(\mathbb{R},+)}{\mathbb{R}} \to G \subset GL(n,\mathbb{R})$ is called a "one-parameter subgp of G if γ is a continuous group homomorphism

$\gamma : \mathbb{R} \to M(n,\mathbb{R})$

Prob 4.1.1 (c) $\gamma(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in M(n,\mathbb{R})$ ① $\det \gamma(t) = 1$

$\gamma(t+s) = \begin{pmatrix} e^{t+s} & 0 \\ 0 & e^{-(t+s)} \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} = \gamma(t)\gamma(s)$ $\gamma : \mathbb{R} \to GL(n,\mathbb{R})$. conti. homo.

4.2 The exponential map in dimension 1.

Note $GL(1,\mathbb{R}) \approx \mathbb{R}\setminus\{0\}$

$[a] \longleftrightarrow a$ $\qquad GL(1,\mathbb{R}) \approx \mathbb{R}\setminus\{0\}$.

Prob 4.2.2

(1) If $\gamma : \mathbb{R} \to GL(1,\mathbb{R})$: one-para. subgp,

then $\gamma(t) = e^{at}$ for $\forall t \in \mathbb{R}$.

Pf with $\gamma : \mathbb{R} \to \mathbb{R}\setminus\{0\}$. Assume $\gamma : \mathbb{R} \to GL(1,\mathbb{R}) = \mathbb{R}\setminus\{0\}$ is a one-parameter subgroup of $GL(1,\mathbb{R})$ and set $\gamma(1) = b > 0 (b > 0$ is because homomorphism maps identity to identity; $\gamma(0) = 1$ and γ is continuous $\gamma(0)$ and $\gamma(1)$ have the same sign by the intermediate value theorem).

(1) For each $a \in \mathbb{R}$, $\gamma(ma) = \gamma(a + \cdots + a) = \gamma(a)\cdots\gamma(a) = (\gamma(a))^m$ for all positive integer m.
(2) Using the result in (1), $\gamma(m) = b^m$ for all positive integer m.
(3) Use $\gamma(m)\gamma(-m) = \gamma(m + (-m)) = \gamma(0) = I_1 = 1$ to show that $\gamma(-m) = (\gamma(m))^{-1} = b^{-m}$ for all positive integer m.
(4) For any $n, m \in \mathbb{N}$, since $n/m > 0$, $\gamma(n/m) > 0$ by continuity and the intermediate value theorem. Then,
$$b^n = \gamma(n) = \gamma\left(m \cdot \frac{n}{m}\right) = \left(\gamma\left(\frac{n}{m}\right)\right)^m \implies \gamma\left(\frac{n}{m}\right) = b^{\frac{n}{m}}.$$
Combining this with (3), we have $\gamma(t) = b^t$ for all $t \in \mathbb{Q}$.
(5) Since γ is continuous, for each $t \in \mathbb{R}$, we can take a sequence of rational numbers $\{r_n\}$ such that $r_n \to t$ as $n \to \infty$. Then,
$$\gamma(t) = \gamma\left(\lim_{n\to\infty} t_n\right) = \lim_{n\to\infty} \gamma(t_n) = \lim_{n\to\infty} b^{t_n} = b^t.$$
(6) Finally, set $a = \log b \in \mathbb{R}$, then we have
$$\gamma(t) = b^t = e^{(\log b)t} = e^{at},$$
and since $\gamma(t)$ is smooth and $\gamma(0) = I_1$, we have $\gamma'(0) = a \in L(G)$.

$\gamma(t)_{n}$ General

(2) If $\gamma : \mathbb{R} \to GL(n,\mathbb{R})$: one-para. subgp,

then $\gamma(t) = e^{tA}$ for $\forall t \in \mathbb{R}$.

It make sense to multiply a matrix A by a scalar t,

but what does it mean e^{matrix} ?

4.3. Calculating the matrix exponential

Note $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Def $\exp : M(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$ is defined by a series

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \underbrace{\frac{1}{k!}A^k}_{=:\frac{A^k}{k!}}$$

Rmk e^A is well-defined for every $A \in M(n,\mathbb{R})$.

pf) Let $\|\cdot\|$ a norm on $M(n,\mathbb{R})$ as

$$\|A\| = n \cdot \max_{i,j} |A_{ij}| \quad \text{where} \quad A = [A_{ij}] \in M(n,\mathbb{R}).$$

Using $\|BC\| \le \|B\| \cdot \|C\|$

$$\boxed{\begin{aligned} \|BC\| &:= n \cdot \max \left| \sum b_{ik} c_{kj} \right| \\ &\le n \cdot \sum_{k=0}^{n} \frac{\|B\|}{n} \cdot \frac{\|C\|}{n} \\ &= \|B\| \|C\| \end{aligned}}$$

$$\sum_{k=0}^{\infty} \left\| \frac{A^k}{k!} \right\| = \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = \underbrace{e^{\|A\|}}_{\text{real exp.}} < \infty.$$

Prob 4.3.1 Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find e^{tA}

$\boxed{\begin{array}{l} \circ \text{ Find } e^{tA} \\ \circ\ \gamma'(0) = A \\ \circ \det(e^{tA}) = e^{tr(tA)} \end{array}}$

$$tA = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}, \quad (tA)^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ k \ge 2$$

$$e^{tA} = I + tA + \frac{(tA)^2}{2!} + \cdots = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

$$\det(e^{tA}) = 1 \times 1 = e^{0+0} = e^{tr(tA)}.$$

Prob 4.3.3 Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find e^{tA}.

diff. signs.

$\circ\ (tA)^2 = \begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix}$ $\quad (tA)^3 = \begin{bmatrix} 0 & t^3 \\ -t^3 & 0 \end{bmatrix}$

$(tA)^4 = \begin{bmatrix} t^4 & 0 \\ 0 & t^4 \end{bmatrix} = t^4 \cdot I \quad \cdots$

$$e^{tA} = \begin{bmatrix} \underbrace{1 + \frac{-t^2}{2!} + \frac{t^4}{4!} - \cdots}_{=\cos t} & -\sin t \\ \underbrace{t + \frac{-t^3}{3!} + \frac{t^5}{5!} - \cdots}_{= \sin t} & \cos t \end{bmatrix}$$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$\ominus$

$\circ\ \gamma'(t) = \begin{bmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{bmatrix} = A \cdot \gamma(t)\ \ \&\ \gamma(0) = I$

$\Rightarrow \gamma'(0) = A.$

Prob 4.3.5 $\quad e^A = I \not\Rightarrow A = O_n$ when $n \geq 2$.

From Prob 4.3.3

For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\quad e^{tA} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$.

When $t = 2\pi$, $\quad e^{2\pi A} = I$ but $2\pi A = \begin{bmatrix} 0 & -2\pi \\ 2\pi & 0 \end{bmatrix}$

is not a zero ma.

Prob 4.3.8 ?

4.4 Properties of the matrix exponential

Proposition 4.1 (Algebraic properties of the exponential function). *Let* $A, B \in M(n,\mathbb{R})$.

(1) $\exp(O_n) = I_n$. $e^{O_n} = I_n$

(2) *If* $AB = BA$, *then* $\exp(A+B) = \exp(A)\exp(B)$. $e^{A+B} = e^A \cdot e^B$

(3) *For any* $A \in M(n,\mathbb{R})$, $\exp(A)$ *is invertible and* $(\exp(A))^{-1} = \exp(-A)$. $(e^A)^{-1} = e^{-A}$

(4) *For any* $A \in M(n,\mathbb{R})$, $\exp(A^T) = (\exp(A))^T$. $(e^A)^T = e^{A^T}$

(5) *For any* $A \in M(n,\mathbb{R})$, $P \in GL(n,\mathbb{R})$, $\exp(PAP^{-1}) = P\exp(A)P^{-1}$. $e^{PAP^{-1}} = Pe^AP^{-1}$

Proof. The proof is basic but somewhat lengthy calculations, I'll omit the details here. 생략 □

$$e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

$$\boxed{[e^{tA}]_{ij} = \sum_{k=0}^{\infty} \frac{[A^k]_{ij}\, t^k}{k!}}$$

abs. conv.

$$\left|[A^k]_{ij}\right| \le \frac{\|A\|^k}{n}.$$

Using the fact that e^A : abs. conv

$$\therefore \frac{d}{dt} e^{tA} \overset{*}{=} \frac{d}{dt} \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$$

diff. term by term. $= \displaystyle\sum_{k=0}^{\infty} \frac{k t^{k-1} A^k}{k!} = Ae^{tA}.$

$$\therefore \gamma(t) = e^{tA} \rightarrow \gamma'(t) = A\gamma(t)$$

$\therefore$ ODE $\begin{cases} y'(t) = Ay(t) \\ y(0) = y_0 \end{cases} \xrightarrow{\text{sol}} y(t) = e^{tA}\cdot y_0.$

Prob 4.4.4

① $e^{O_n} = I_n$

② $e^{A} \in GL(n,\mathbb{R})$ for $\forall A \in M(n,\mathbb{R})$.

③ $e^{(t+s)A} = e^{tA} \cdot e^{sA}$

Pf) $e^{O_n} = I_n + O_n + \cdots = I_n$ (by def).

$$e^{A} \cdot e^{-A} \underset{\uparrow}{=} e^{A-A} = e^{O_n} = I_n$$

A, −A : commute.

$\therefore \gamma(t) = e^{tA}$ is a one-parameter subgp of $GL(n,\mathbb{R})$ & smooth, $\gamma(0) = I_n$

$\therefore A \in L(G)$.

$$L(GL(n,\mathbb{R})) \overset{\supset}{\underset{\subset \text{ by def.}}{=}} M(n,\mathbb{R}).$$

P. 4.3.3

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \gamma(t) = e^{tA} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$\gamma(0) = \gamma(2\pi)$$

4.5 Using exp to determine $L(G)$.

We will prove a very useful theorem $\det(\underbrace{e^{A}}_{\text{matrix exp}}) = \underbrace{e^{tr(A)}}_{\text{real exp}}$.

It can be expressed the following commutative diagram.

$$\begin{tikzcd}
GL(n,\mathbb{R}) \arrow[rr, "\det"', "\| \; F"] & & GL(1,\mathbb{R}) \\
M(n,\mathbb{R}) \arrow[u, "\exp"] \arrow[rr, "tr", "\| \; dF = dF_I"'] \arrow[urr, phantom, "\circlearrowright"] & & M(1,\mathbb{R}) \arrow[u, "\exp"']
\end{tikzcd}$$

<u>Examples</u> ① $A = \operatorname{diag}(a_1, \dots, a_n)$ $\quad A^K = \operatorname{diag}(a_1^K, \dots, a_n^K)$

e^A: diag & $[e^A]_{ii} = \sum_{K=0}^{\infty} \frac{[A^K]_{ii} = a_i^K}{K!} = e^{a_i}$

Proof. (1) For diagonal matrix $A = \operatorname{diag}(a_1, \dots, a_n)$, we have

$$e^A = \operatorname{diag}(e^{a_1}, \dots, e^{a_n}) \implies \det(e^A) = e^{a_1} \cdots e^{a_n} = e^{\operatorname{tr}(A)}.$$

② A:
For a diagonalizable matrix $A = PDP^{-1}$, where D is a diagonal matrix and $P \in GL(n, \mathbb{R})$, we have

$$\det(e^A) = \det\left(e^{PDP^{-1}}\right) = \det(Pe^D P^{-1}) = \det(e^D) = e^{\operatorname{tr}(D)} = e^{\operatorname{tr}(A)}.$$

③ $A \in M(n,\mathbb{R})$

$$\begin{tikzcd}
e^{tA} \in GL(n,\mathbb{R}) \arrow[rr, "\det : \text{conti}", "\| \; F"'] & & GL(1,\mathbb{R}) \ni \det(e^{tA}) \\
tA \in M(n,\mathbb{R}) \arrow[u, "\exp"] \arrow[rr, "tr", "\| \; dF = dF_I"'] \arrow[urr, phantom, "\circlearrowright"] & & M(1,\mathbb{R}) \arrow[u, "\exp"']
\end{tikzcd}$$

$t \in \mathbb{R}$, $A \in M(n,\mathbb{R})$

set $\gamma(t) = \det(e^{tA})$

- $\gamma : \mathbb{R} \to GL(n,\mathbb{R})$ well-def
- γ: conti (smooth) ($\exp, \det$: conti (smooth))
- $\gamma(t+s) = \det(e^{(t+s)A})$

$= \det(e^{tA} e^{sA})$

$(\because (tA)(sA) = (sA)(tA))$

$= \det(e^{tA}) \det(e^{sA})$

$= \gamma(t)\gamma(s)$

$\therefore\ \gamma(t)$ is a one-parameter subgp of $GL(n,\mathbb{R})$

$\gamma(t) = e^{bt}$ for some $b \in \mathbb{R}$. (by Prob. 4.2.2)

$F = \det$, $\gamma(t) = F(e^{tA}) = e^{bt} \quad \forall t \in \mathbb{R}$

$\xrightarrow[\text{diff}]{}$ $\circ\ \left.\frac{d}{dt}\right|_{t=0} F(\underset{\sim}{e^{tA}}) = dF\Big(\underbrace{\left.\frac{d}{dt}\right|_{t=0} e^{tA}}_{\overset{\shortparallel}{A}}\Big) = dF(A)$

$e^{0A} = I_n$ & smooth $\qquad\qquad = \mathrm{tr}(A)$

$\circ\ \left.\frac{d}{dt}\right|_{t=0} e^{bt} = b$. $\quad \therefore\ \det(e^A) = e^b = e^{\mathrm{tr}(A)}$ ▨

Most important thm.

Thm 4.4 $\quad G$: Lie subgp of $GL(n,\mathbb{R})$, $L(G) = T_{I_n}G$

Then, $\quad A \in L(G) \Rightarrow e^A \in G$.

i.e. $\quad L(G) = \{ A \in M(n,\mathbb{R}) \mid e^{tA} \in G \text{ for } \forall t \in \mathbb{R} \}$

Not prove, just examples.

$\circ\ G = O(n,\mathbb{R})$: $\forall A \in L(G) = \{ A \in M(n,\mathbb{R}) : A^T = -A \}$.

For a smooth curve $\gamma : \mathbb{R} \to G$ s.t.

$\gamma(0) = I_n$, $\gamma'(0) = A$,

$\gamma(t)^T \gamma(t) = I \overset{\text{di}}{\Rightarrow} \gamma'(t)^T \gamma(t) + \gamma(t)^T \gamma'(t) = 0$

$\overset{t=0}{\Rightarrow} \quad A^T I + I^T A = 0 \quad : A^T = -A$.

$\Rightarrow \det(e^A) = \det($

Remark. *By using the geometric series:*

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1,$$

we can find the taylor series of $\log(1+x)$ *at* $x = 0$ *as follows:*

$$\log(1+x) = \int_0^x \frac{1}{1+t}dt = \int_0^x \sum_{k=0}^{\infty}(-1)^k t^k dt = \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad |x| < 1.$$

Then,

$$\log(x) = \log(1 + (x-1)) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots, \quad |x-1| < 1.$$

Definition. *For* $M \in GL(n, \mathbb{R})$*, the natural* ***logarithm*** *of* M *is defined as*

$$\log(M) := (M - I_n) - \frac{1}{2}(M - I_n)^2 + \frac{1}{3}(M - I_n)^3 - \frac{1}{4}(M - I_n)^4 + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(M - I_n)^k,$$

provided that $\|M - I_n\| < 1$*(using* $\|(M - I_n)^m\| \leq \|M - I_n\|^m$*, we can show if* $\|M - I_n\| < 1$*, the matrix-valued series will converge)*

Proposition 4.8 (Properties of the logarithm function)**.** *For* $M \in GL(n, \mathbb{R})$ *and* $A \in M(n, \mathbb{R})$*, we have the following:*

(1) The series for $\log(M)$ *converges for* $\|M - I_n\| < 1$*.*
(2) When A *is near* O_n*,* $\log(e^A) = A$*. And when* M *is near* I_n*,* $e^{\log(M)} = M$*.*
(3) If M_1 *and* M_2 *are near* I_n *and* $\log(M_1)$ *and* $\log(M_2)$ *commute, then*

$$\log(M_1 M_2) = \log(M_1) + \log(M_2).$$

Proof. (3) Since M_1 and M_2 are near I_n, we can write

$$X = \log(M_1), \quad Y = \log(M_2),$$

where X and Y are near O_n. Since X and Y commute, by Prop4.1(2),

$$M_1 M_2 = e^X e^Y = e^{X+Y}.$$

Since exp is invertible, and therefore one-to-one near $O_n \in M(n, \mathbb{R})$,

$$\log(M_1 M_2) = \log(e^{X+Y}) = X + Y = \log(M_1) + \log(M_2).$$

□