Chap 5 Lie algebras

<5.1> Lie algebras

Recall $L(G)$: Tangent space of G at identity.

$L(G) \leq M(n, \mathbb{R})$ as a vector space & $\dim L(G) = \dim G$

$(G \subset M(n, \mathbb{R}))$

$\Rightarrow$ Different Lie groups can produce isomorphic tangent space. (as a vector space)

"Add more structure to enrich tangent space" $\Rightarrow$ Can we classify Lie groups using tangent spaces?

Def Lie algebra

$(L, +, \cdot)$ be a (real) vector space. If $[\cdot, \cdot]: L \times L \to L$ satisfies

① closed ② anti-symmetric ③ bilinear ④ Jacobi indentity,

$[v, w] = -[w, v]$ $\quad$ $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$

then $(L, +, \cdot, [\cdot, \cdot])$ is called a Lie algebra, and $[\cdot, \cdot]$ is called a Lie bracket.

(Ex) $[A, B] = AB - BA$ (commutator) is a Lie bracket to $M(n, \mathbb{R})$. (cf) Abelian $\Leftrightarrow \operatorname{Im}[\cdot, \cdot] = \{0\}$.

Actually, it makes many matrix groups a Lie algrebra.

For example, tangent space of $[SL(n, \mathbb{R}), O(n, \mathbb{R}), GL(n, \mathbb{R}) \cdots]$

(Ex) cross product on $\mathbb{R}^3$.

Def Lie algebra homomorphism / isomorphism

Preserves $+, \cdot, [\cdot, \cdot]$. If 1-1 corr., then isomorphism.

(Ex) $L(O(3, \mathbb{R})) \simeq \mathbb{R}^3$ as Lie algebra. $(x, y, z) \mapsto \begin{pmatrix} 0 & x & -y \\ x & 0 & -z \\ y & z & 0 \end{pmatrix}$.

$= \{B^T = -B\}$.

<5.2> Adjoint maps – big 'A' and small 'a'

Motivation for commutator. (as a Lie bracket)

Recall matrix conjugate: $X \mapsto MXM^{-1}$. It is linear & invertible.

> **Recall** $GL(n,\mathbb{R})$: invertible $M(n,\mathbb{R})$
>
> $GL(V)$: invertible linear map $V \to V$

Def 'Adjoint map

$$Ad : GL(n,\mathbb{R}) \to GL(\mathbb{R}^{n^2}) \; \left(\simeq GL(M(n,\mathbb{R}))\right)$$

$Ad(M) : X \mapsto MXM^{-1}$, i.e. $Ad(M)(X) = MXM^{-1}$.

[In general, for Lie group G, $Ad : G \to Aut(L(G))$, Ad_g is a derivative of conjugate]

prop ① $Ad(M)$ is linear, invertible, preserves $[\,,\,]$, isomorphic

$\Rightarrow$ $Ad(M)$ is Lie algebra isomorphism (automorphism) on $M(n,\mathbb{R})$.

② Ad is a group homomorphism.

$$Ad(M_1M_2) = Ad(M_1) \circ Ad(M_2)$$

Def 'adjoint map

'ad' is defined as 'Ad''s derivative at identity.

$ad : L(G_1) \to L(G_2)$ where $G_1 = GL(n,\mathbb{R})$, $G_2 = GL(\mathbb{R}^{n^2})$. $\Rightarrow$ $L(G_1) = M(n,\mathbb{R})$, $L(G_2) = M(n^2,\mathbb{R})$

~~Let $\gamma_1 : (-\epsilon,\epsilon) \to L(G_1) = M(n,\mathbb{R})$ Define $\gamma_2(t) =$~~

To compute $ad(A)$ for $A \in L(G_1) = M(n,\mathbb{R})$, by definition of derivative,

① Take $\gamma_1(t) = e^{tA}$ so that $\gamma_1'(0) = A$.

② $\gamma_2(t) = Ad(\gamma_1(t))$. Note that $e^{tA} \in GL(n,\mathbb{R})$.

③ $ad(A) = \gamma_2'(0)$.

prop ① $ad(A)(B) = \overbrace{AB - BA}^{[A,B]}$. – Start from $\gamma_2(t)(B) = Ad(\gamma_1(t))(B)$.

② ad is linear & preserves $[\,,\,]$ $\Rightarrow$ Lie algebra homomorphism.

③ It is enough to know $[v_i, v_j]$ for basis $\{v_i\}_i$

④*

$$\begin{tikzcd} GL(n,\mathbb{R}) \arrow[r, "Ad"] & GL(\mathbb{R}^{n^2}) \simeq GL(n^2,\mathbb{R}) \\ M(n,\mathbb{R}) \arrow[u, "\exp"] \arrow[r, "ad"'] & M(n^2,\mathbb{R}) \arrow[u, "\exp"'] \end{tikzcd}$$

(cf) $ad = d(Ad)$

<5.4> Lie theory

Thm*

→ up to Lie group homomorphism

1. If L is a Lie algebra, $\exists!$ simply connected Lie group $G(L)$ s.t.
   $L(G(L)) \simeq L$.
2. G : connected Lie group. Then $G(L(G))$ is called the 'universal cover' of G in the sense that $\exists$ homomorphism $\phi : G(L(G)) \to G$. Furthermore, $\ker \phi \le G(L(G))$ is discrete.

Thm*

1. (prop 5.9) If $G \subseteq GL(n, \mathbb{R})$ is an abelian Lie group, then $L(G) \subseteq M(n, \mathbb{R})$ is abelian, too.
2. (thm 5.11) If G is a connected Lie group and $L(G)$ dimension is $n > 0$,
   is abelian and it's
   then G is also abelian.

Thm*

Let G_1, G_2 be simply connected Lie groups. Let $f : L(G_1) \to L(G_2)$ be a Lie algebra homomorphism. Then $\exists$ Lie group homomorphism $F : G_1 \to G_2$ s.t. $dF = f$. (cf) Thm 3.9*

Thm*

If G_1, G_2 are simply connected Lie groups, and $L(G_1) \simeq L(G_2)$, then
in Lie alg.
$G_1 \simeq G_2$.
in Lie group