Chap 6: Matrix groups over other fields.

$\boxed{1}$ Definition of field: set with two operations (addition $+$, multiplication $*$)

satisfying ① $(F, +)$ is abelian group with identity 0.

② $(F/\{0\}, *)$ is abelian group with identity 1

③ Distribution law $a(b+c) = ab + ac$

ex) $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$

As we have done in real numbers, we investigate the matrix group over $\mathbb{F}$ and their Lie algebra.

$\boxed{2}$ Unitary group.

We have defined the Unitary group $U(n, \mathbb{C}) = \{M \in M(n, \mathbb{C}) : \overline{M}^T M = I_n\}$

and special unitary group $SU(n, \mathbb{C}) = \{M \in U(n, \mathbb{C}) : \det(M) = 1\}$

Def Matrix $M \in M(n, \mathbb{C})$ is Hermitian iff $M^* := \overline{M}^T = M$.

Def $x \cdot y = \overline{x_1} y_1 + \cdots + \overline{x_n} y_n$ is called Hermitian inner product.

Def $|v| = \sqrt{v \cdot v}$ is the length of vector $v \in \mathbb{C}^n$.

Now, we want to associate a tangent space at identity to each of these matrix groups over the $\mathbb{C}$.

Preliminaries: differentiation of complex vectors & matrices

Then now we may define a smooth curve passing through the identity in $GL(n, \mathbb{C})$

$$L(G) = \{ \gamma'(0) : \gamma: \mathbb{R} \to G,\ \gamma \text{ smooth},\ \gamma(0) = I_n \} \text{ for } G \leq GL(n, \mathbb{C})$$

The matrix exponential can be defined in similar manner,

$$\exp(A) = I + A + \frac{A^2}{2!} + \cdots \qquad \text{for } A \in M(n, \mathbb{C})$$

and $\exp(\bar{A}) = \overline{\exp(A)}$

Problem 6.2.6 $\sim$ 6.2.8 let $G = SU(n, \mathbb{C})$, $\gamma: \mathbb{R} \to G$ is a smooth curve passing identity.

(a) Show that $\gamma(t) \in U(n, \mathbb{C})$ implies $\overline{\gamma'(0)}^T + \gamma'(0) = O_n$.

pf) $\gamma(t) \in U(n, \mathbb{C}) \Rightarrow \overline{\gamma(t)}^T \gamma(t) = I_n. \Rightarrow \overline{\gamma'(t)}^T \gamma(t) + \overline{\gamma(t)}^T \gamma'(t) = O_n$

$\Rightarrow \overline{\gamma'(0)}^T \gamma(0) + \overline{\gamma(0)}^T \gamma'(0) = O_n$

$\Rightarrow \overline{\gamma'(0)}^T + \gamma'(0) = O_n$.

(b) Suppose $n = 2$ and $\gamma'(0) = \begin{pmatrix} a'(0) & b'(0) \\ c'(0) & d'(0) \end{pmatrix}$ with $a'(0) = a_1 + i a_2$ etc.

What does (a) tell you?

ans) $\overline{\gamma'(0)}^T + \gamma'(0) = O_n$ implies $\begin{pmatrix} a'(0) + \overline{a'(0)} & b'(0) + \overline{c'(0)} \\ c'(0) + \overline{b'(0)} & d'(0) + \overline{d'(0)} \end{pmatrix} = O_n \Rightarrow$ $a'(0) + \overline{a'(0)} = 0$, $d'(0) + \overline{d'(0)} = 0$, $b'(0) + \overline{c'(0)} = 0$

also, $\det(\gamma(t)) = 1$ implies $tr(\gamma'(0)) = 0$.

$$\left(\because 0 = \frac{d}{dt} \det(\gamma(t))\Big|_{t=0} = d(\det)\big|_{\gamma(t)} \circ \gamma'(t)\big|_{t=0} = tr(\gamma'(0))\right)$$

$\Rightarrow a'(0) + d'(0) = 0$. This implies a, b, c, d has the form

$a = ui$, $d = -ui$, $b = v + wi$, $c = -v + wi$ $\Rightarrow L(G) \subseteq W := \left\{ \begin{bmatrix} ui & v+wi \\ -v+wi & -ui \end{bmatrix} \middle| u, v, w \in \mathbb{R} \right\}$

(c) We have shown this in (b).

(d)(e), W is closed in addition and real scalar multiplication,

but not in complex scalar multiplication, $\Rightarrow$ $L(G)$ is only real vector space.

6.2.8 Here we show that $L(G) = W$.

let $\sigma_1, \sigma_2, \sigma_3$ be Pauli spin matrix, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Note: $\exp(it\sigma_1) = \left(I - \frac{t^2\sigma_1^2}{2!} + \frac{t^4\sigma_1^4}{4!} \cdots\right) + i\left(\frac{t\sigma_1}{1!} - \frac{t^3\sigma_1^3}{3!} + \cdots\right)$

$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} \cdots\right) I + i\left(\frac{t}{1!} - \frac{t^3}{3!} + \cdots\right)\sigma_1 = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}$

$\rightarrow$

Similarly, $\exp(it\sigma_2) = (\cos t) I + i\sin t\,\sigma_2 = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

$\exp(it\sigma_3) = (\cos t) I + i\sin t\,\sigma_3 = \begin{pmatrix} \cos t + i\sin t & 0 \\ 0 & \cos t - i\sin t \end{pmatrix} = 1$

and all $\exp(it\sigma_j) \in SU(2, \mathbb{C}) = G \Rightarrow i\sigma_j \in L(G)$ $\quad j = 1, 2, 3$

$\Rightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in L(G) \Rightarrow W \in L(G)$

($\because$ $L(G)$ is closed under addition)

6.2.10.

a. Assume $A \in M(n, \mathbb{C})$ and $\bar{A}^T + A = 0_n$, explain why $\bar{A}^T A = A\bar{A}^T$

pf) $\bar{A}^T A = -AA = A(-A) = A\bar{A}^T$

b. Let $G = U(n, \mathbb{C})$, explain why $L(G) = \{A \in M(n, \mathbb{C}) : \bar{A}^T + A = 0_n\}$

pf) $\subseteq$ is already shown.

for b, pick $A \in \{A \in M(n,\mathbb{C}) : \bar{A}^T + A = O_n\}$

Claim: $\gamma(t) = \exp(tA)$ satisfies $\gamma(t) \in G = U(n,\mathbb{C})$ and $\underbrace{\gamma'(0) = A}_{= \text{o.k.}}$

pf) $\overline{\gamma(t)}^T \gamma(t) = \overline{\exp(tA)}^T \exp(tA) = \exp(t\bar{A}^T)\exp(tA)$

$= \exp(t(\bar{A}^T + A)) = \exp(tO_n) = I_n$, $\therefore \gamma(t) \in U(n,\mathbb{C})$.

($\because$ by a, $\bar{A}^T$ and A commute)

$\therefore L(G) = \{A \in M(n,\mathbb{C}) : \bar{A}^T + A = O_n\}$

c. same, $\det(\exp(tA)) = 1 \Leftrightarrow \exp(\mathrm{tr}(tA)) = 1 \Leftrightarrow \mathrm{tr}(tA) = 0 \Leftrightarrow \mathrm{tr}(A) = 0$.

d. ① $\overline{[A,B]}^T = B^* A^* - A^* B^* = BA - AB = -[A,B]$,

② $\mathrm{tr}([A,B]) = \mathrm{tr}(BA) - \mathrm{tr}(AB) = 0$.

$\rightarrow$ This implies the commutator is still a valid Lie algebra.

6.2.12. Show that $L(G)$ is real vector space.

$A \in L(G) \Rightarrow \exists \gamma : \mathbb{R} \to G$ s.t $\gamma'(0) = A$, $\gamma(0) = I_n$

$\Rightarrow$ let $\tilde{\gamma}(t) = \gamma(ct)$, then $\tilde{\gamma}'(0) = c\gamma'(0) = cA \Rightarrow cA \in L(G)$ for $c \in \mathbb{R}$.

If $c \in \mathbb{C}$, $\tilde{\gamma}$ may undefined.

Show that

6.2.13 ① $L(GL(n,\mathbb{C})) = M(n,\mathbb{C})$ $\Rightarrow$ given $A \in M(n,\mathbb{C})$, consider $\gamma = e^{At}$

② $L(SL(n,\mathbb{C})) = \{A \in M(n,\mathbb{C}),\ \mathrm{tr}(A) = 0\}$

$\Rightarrow$ given $A \in M(n,\mathbb{C})$ s.t $\mathrm{tr}(A) = 0$, $\gamma = e^{At} \Rightarrow \det(\gamma) = e^{\mathrm{tr}(A)t} = 1$.

③ closed under complex scalar multiplication & matrix braket is straightforward.

3 Finite fields.

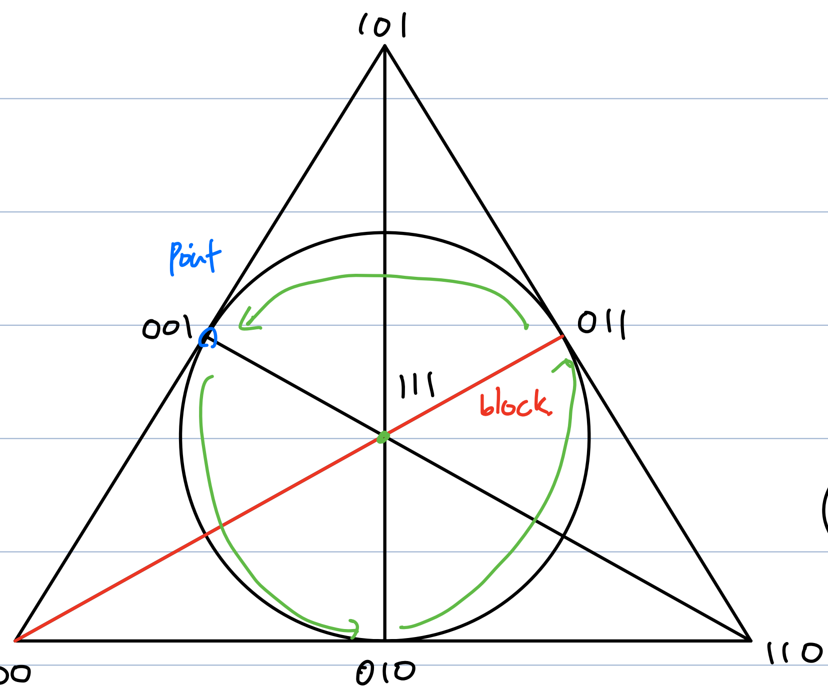
e.g. $\mathbb{F}_2$, $\mathbb{F}_p$,

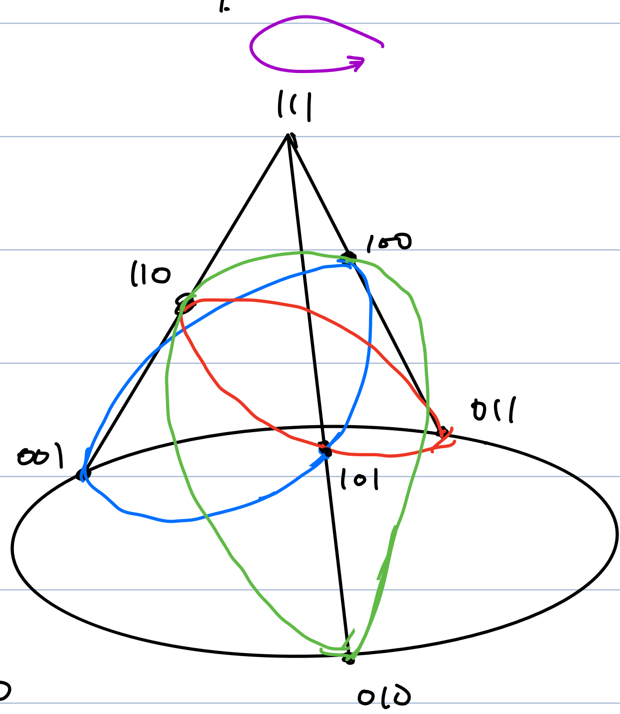
Goal: Investigating $G = SL(3, \mathbb{F}_2)$ as a group of symmetries of v.s. $V = (\mathbb{F}_2)^3$.

and related geometry called a symmetric design.

Subspace of V: ① Seven one-dimensional $\{0, v\}$

② Seven two-dimensional $\{0, v, w, v+w\}$





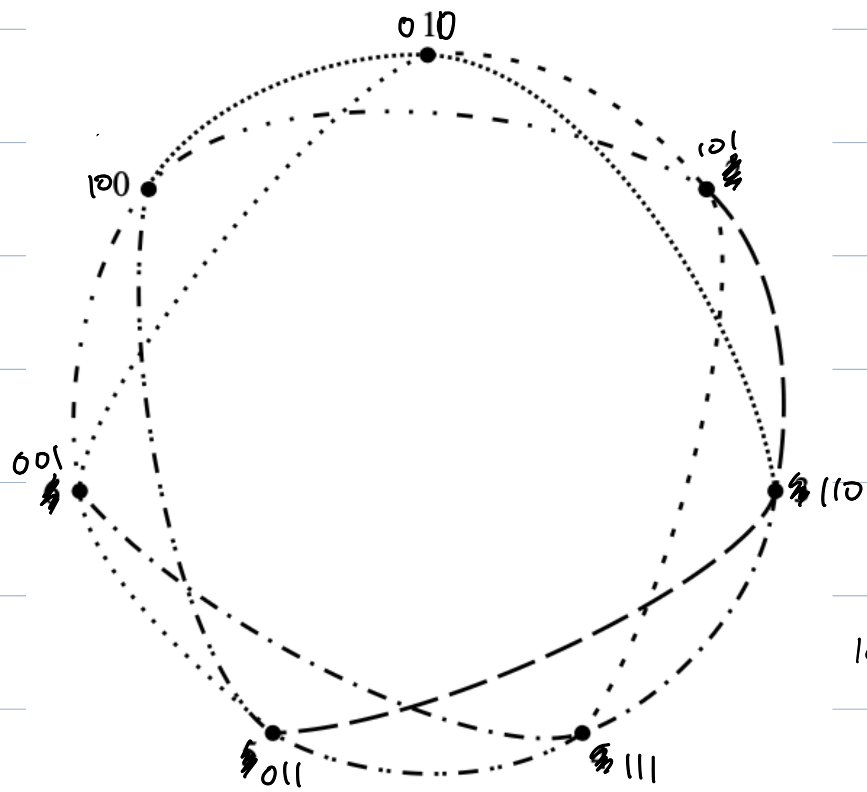
6.3.7 $M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in SL(3, \mathbb{F}_2)$, $M_1^2 = I_3$.

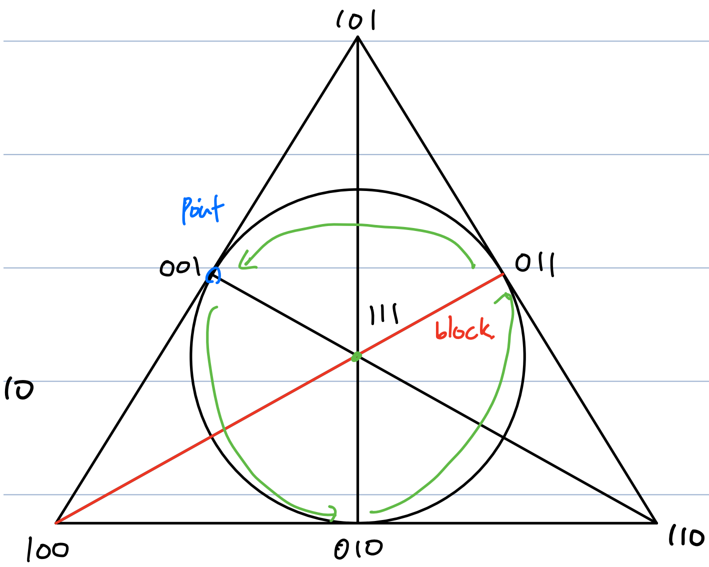
$M_1 \Leftrightarrow$ reflection with respect to line 100—111—011 (red one)

6.3.8 $M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in SL(3, \mathbb{F}_2)$. $M_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $M_1^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$

$$\begin{tikzcd} 001 \arrow[dr] & & 011 \arrow[ll] \\ & 010 \arrow[ur] & \end{tikzcd}, \quad \begin{tikzcd} & 101 \arrow[dl] & \\ 100 \arrow[rr] & & 110 \arrow[ul] \end{tikzcd} \quad 111 \to 111$$

6.3.9 $M_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$; $100 \to 010 \to 101 \to 110 \to 111 \to 011 \to 001$





(7,3,1) design.

In general, we call this (v, k, λ) symmetric design.

→ v points, v blocks, k points in one block,

Two distinct block have exactly λ point in common,

Two points belongs to λ block.

$\boxed{4}$ Finite groups of Lie types.

Intrinsically, finite groups are discrete ⇒ No calculus can be made.

⇒ No tangent spaces & Lie algebra.

(Recall that, chap 5 devoted to capture group's info from its tangent space)

Then, we start from Lie algebra.

Lie algebra L: v.s. with bracket op. satisfies
(with scalar field $\mathbb{F}$)

- $v, w \in L \Rightarrow [v,w] \in L$
- $[w,v] = -[v,w]$
- bilinearity
- Jacobi identity.

Def $\mathrm{Aut}(L) = \{T : L \to L : T$ is Lie algebra automorphism$\}$.

Then $\mathrm{Aut}(L)$ is a group under composition.

Now we may build $G(L, \mathbb{F}) \leq \mathrm{Aut}(L)$ (which will be G)

→ L acts as tangent space if $\mathbb{F}$ admits differentiation.

Theorem 6.5.2* (Chevalley) *Assume L is a finite-dimensional simple Lie algebra over the complex numbers. Then L has a basis $\{X_1, X_2, \ldots, X_n\}$ for which all the "structure constants" are integers; that is, $[X_i, X_j] = \sum a_{ijk} X_k$ and the a_{ijk} are integers, for all $i, j, k = 1, \ldots, n$.*

→ let it $\mathcal{B}$

Define $\mathrm{ad}(X) : L \to L$ by $\mathrm{ad}(X)(Y) = [X, Y]$

⇒ Prop. If $X \in \mathcal{B}$, then matrix representation of $\mathrm{ad}(X)$ is
- diagonal
- strictly upper/lower triangular.

$\Rightarrow \exp(t\,\mathrm{ad}(x))$ can be defined. for every $X \in \mathcal{B}$. and it preserves bracket op.

$\Rightarrow$ We define $G(L, \mathbb{F}) = \langle M(t, X) = \exp(t\,\mathrm{ad}(X)) : t \in \mathbb{F},\ X \in \mathcal{B} \rangle$

(generated by $\exp(t\,\mathrm{ad}(X))$)

This $G(L, \mathbb{F})$ is called Chevalley group of adjoint type ass. with L over $\mathbb{F}$.