8.3. ~~Ap~~ Adaptive Control of Linear Systems With Full State Feedback.

- Consider the adaptive control of n^{th}-order linear systems

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = u \qquad (1)$$

- state components $y, \dot{y}, \cdots, y^{(n-1)}$ are measurable
- assume coefficient vector $a = \begin{pmatrix} a_n \\ \vdots \\ a_1 \\ a_0 \end{pmatrix}$ is unknown, the sign of a_n is known

Example: An example of such systems is the dynamics of a mass-spring-damper system

$$m\ddot{y} + c\dot{y} + ky = u.$$

Objective: make y closely track the response of a stable reference model

$$\alpha_n y_m^{(n)} + \alpha_{n-1} y_m^{(n-1)} + \cdots + \alpha_0 y_m = r(t) \qquad (2).$$

with $r(t)$ being a bounded reference signal

Choice of control law:

- Define a signal $z(t) = y_m^{(n)} - \beta_{n-1} e^{(n-1)} - \cdots - \beta_0 e \qquad (3).$

with $\beta_1, \cdots, \beta_n$ being potive constants chosen s.t. $p^n + \beta_{n-1} p^{n-1} + \cdots + \beta_0$ is a stable (Hurwitz) polynomial. i.e $1 > \beta_{n-1} > \cdots > \beta_0 > 0$.

- Adding $(-a_n z(t))$ to both sides of (1) and rearranging. the plant dynamics as

$$a_n[y^{(n)} - z] = u - a_n z - a_{n-1} y^{(n-1)} - \cdots - a_0 y \qquad (4)$$

- Choose the control law to be

$$u = \hat{a}_n z + \hat{a}_{n-1} y^{(n-1)} + \cdots + \hat{a}_0 y = v^T(t)\,\hat{a}(t) \quad (5). \qquad \text{where } v(t) = \begin{pmatrix} z(t) \\ y^{(n-1)} \\ \vdots \\ \dot{y} \\ y \end{pmatrix}$$

$$\hat{a}(t) = \begin{pmatrix} \hat{a}_n \\ \vdots \\ \hat{a}_0 \end{pmatrix} \rightarrow \text{estimated parameter vector}$$

$\Longrightarrow$ The tracking error $e = y - y_m$ satisfies the closed-loop dynamics

$$a_n[e^{(n)} + \beta_{n-1} e^{(n-1)} + \cdots + \beta_0 e] = v^T(t)\,\tilde{a}(t), \quad \text{where } \tilde{a} = \hat{a} - a. \qquad (6)$$

[Substituting (3), (5) into (4), then

$$a_n[y^{(n)} - y_m^{(n)} + \beta_{n-1} e^{(n-1)} + \cdots + \beta_0 e]$$

$$= \hat{a}_n z + \hat{a}_{n-1} y^{(n-1)} + \cdots + \hat{a}_0 y - a_n z - a_{n-1} y^{(n-1)} - \cdots - a_0 y,$$

i.e.

$$a_n[e^{(n)} + \beta_{n-1} e^{(n-1)} + \cdots + \beta_0 e] = (\hat{a}_n - a_n) z + (\hat{a}_{n-1} - a_{n-1}) y^{(n-1)} + \cdots + (\hat{a}_0 - a_0) y$$

$$= v^T(t)\,\tilde{a}(t). \quad]$$

Choice of adaptation law

- Rewrite the closed-loop error dynamics (6) in state space form

$$\dot{x} = Ax + b\left[\left(\frac{1}{a_n}\right) v^T \tilde{a}\right]$$

$$e = cx.$$

where
$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdot \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\beta_0 & -\beta_1 & -\beta_2 & \cdots & -\beta_{n-1} \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad c = (1, 0, \cdots 0).$$

- Consider the Lyapunov function candidate

$$V(X,\tilde{a}) = X^T P X + \tilde{a}^T \Gamma^{-1} \tilde{a}$$

where both Γ and P are symmetric positive definite constant matrics, and P satisfies

$$PA + A^T P = -Q, \quad Q = Q^T > 0 \quad \text{for a chosen } Q$$

- The derivative $\dot{V}$ can be computed as

$$\dot{V} = -X^T Q X + 2\tilde{a}^T v b^T P X + 2\tilde{a}^T \Gamma^{-1} \dot{\tilde{a}}$$

$\Longrightarrow$ Therefore, the adaptation law $\dot{\tilde{a}} = -\Gamma v b^T P X$ (7) leads to

$$\dot{V} = -X^T Q X.$$

$\Longrightarrow$ Therefore, with the adaptive controller defined by control law (5) and adaptation law (7), e and its $(n-1)$ derivatives converge to 0.

§8.4. Adaptive Control of Linear Systems with Output Feedback.

- only output measurement, rather than full state feedback
- A linear time-invariant system can be represented by the transfer function

$$W(p) = k_p \frac{Z_p(p)}{R_p(p)}, \quad \text{where } k_p \text{ is called the high-frequency gain.}$$

$$R_p(p) = a_0 + a_1 p + \cdots + a_{n-1} p^{n-1} + p^n$$

$$Z_p(p) = b_0 + b_1 p + \cdots + b_{m-1} p^{m-1} + p^m$$

- In our adaptive control problem the coefficients a_i, b_j, $i = 0, 1, \cdots, n-1$, $j = 0, 1, \cdots, m-1$

and the high frequency gain k_p are all assumed to be unknown.

- The desired performance is assumed to be described by a reference model with transfer function

$$W_m(p) = K_m \frac{Z_m}{R_m}$$

where Z_m, R_m are monic Hurwitz polynomials of degrees n_m and m_m. k_m is positive.

- In our treatment, we will assume $n_m - m_m \geq n - m$

<u>Objective</u> The objective of the design is to determine a control law, and an associated adaptation law, so that the plant output y asymptotically approaches y_m

8.4.1. Linear Systems with Relative Degree One.

When the relative degree is 1, i.e. $m = n-1$. (The degree of denominator is one order higher than that of numerator

<u>Example:</u> • Consider the plant described by

$$y = \frac{k_p (p + b_p)}{p^2 + a_{p1} p + a_{p2}} u$$

and the reference model

$$y_m = \frac{k_m (p + b_m)}{p^2 + a_{m1} p + a_{m2}} r$$

- Let the controller with the control law being

$$u = \alpha_1 z + \frac{\beta_1 p + \beta_2}{p + b_m} y + kr$$

whre $z = \frac{u}{p + b_m}$, $\alpha_1, \beta_1, \beta_2, k$ are controller parameters.

- If we take these parameters to be

$$\alpha_1 = b_p - b_m,$$

$$\beta_1 = \frac{\alpha_{m_1} - \alpha_{p_1}}{k_p},$$

$$\beta_2 = \frac{\alpha_{m_2} - \alpha_{p_2}}{k_p}$$

$$k = \frac{k_m}{k_p}$$

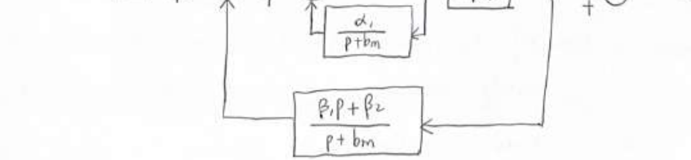
$\Longrightarrow$ one can show that the transfer function from the reference input r to the plant output y is

$$W_{ry} = \frac{k_m (p + b_m)}{p^2 + a_{m_1} p + a_{m_2}} = W_m(p)$$ I have no idea how to calculate this

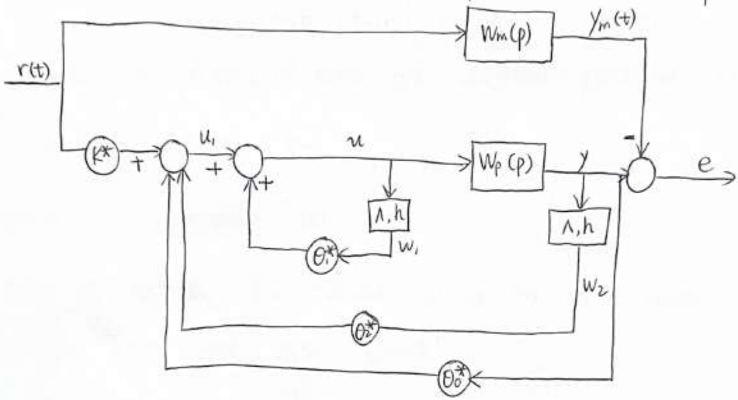
$\Longrightarrow$ perfect tracking is achieved with this control law, i.e. $y(t) = y_m(t)$.

(what does this mean?)

Control System:



- The controller structure for second-order plants can be extended to any plant with relative degree one. The resulting structure is the following, where k^*, θ_1^*, θ_2^* and θ_0^* represents controller parameters.



- The block for generating the filter signal w_1 represents an $(n-1)^{th}$ order dynamics,

$$\dot{w}_1 = \Lambda w_1 + hu$$

where w_1 is an $(n-1)\times 1$ state vector,
Λ is an $(n-1)\times(n-1)$ matrix, and
h is a constant.

$\det(pI - \Lambda) = Z_m(p)$

- The block for generating the $(n-1)\times 1$ vector w_2 has the same dynamics but with y as input, i.e.

$$\dot{w}_2 = \Lambda w_2 + hy$$

- The scalar gain k^* is defined to be $k^* = \frac{k_m}{k_p}$ and is intended to modulate the high-frequency gain of the control system.

- The vector θ_1^* contains $(n-1)$ parameters which intend to cancel the zeros of the plant.

- The vector θ_2^* contains $(n-1)$ parameters which, together with the scaler gain θ_0^* can move the poles of the closed-loop control system to the locations of the reference model poles.

$\Longrightarrow$ To this end, the control law is chosen to be

$$u = k(t) r + \theta_1(t) w_1 + \theta_2(t) w_2 + \theta_0(t) y, \qquad (8)$$

where $k(t), \theta_1(t), \theta_2(t), \theta_0(t)$ are controller parameters to be provided by the adaptation law.

Choice of adaptation law

- Let θ be the $2n \times 1$ vector, w be the $2n \times 1$ vector, i.e.

$$\theta(t) = [k(t) \quad \theta_1(t) \quad \theta_2(t) \quad \theta_0(t)]^T$$

$$w(t) = [r(t) \quad w_1(t) \quad w_2(t) \quad y(t)]^T$$

Then the control law (8) can be compactly written as

$$u = \theta^T(t) w(t) \qquad (9)$$

- Denote the ideal value of θ by θ^*

the error between $\theta(t)$ and θ^* by $\phi(t) = \theta(t) - \theta^*$,

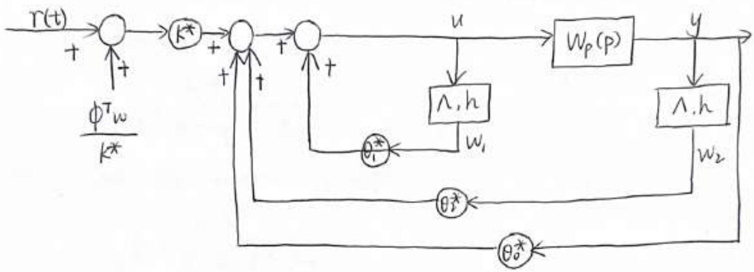
the estimated parameters $\theta(t)$ can be represented as $\theta(t) = \theta^* + \phi(t)$.

Then the control law ~~(8)~~ (9) can be written as

$$u = (\theta^*)^T w + \phi^T(t)\, w.$$

- In order to choose an adaptation law so that the tracking error e converges to zero, we have to first find out how the tracking error is related to the parameter error.

• With the control law given in (9), the control system with variable gains can be represented as follows, with $\frac{\phi^T(t) w}{k^*}$ regarded as an external signal.



the output here must be

$$y(t) = W_m(p) r + W_m(p) \cdot \frac{\phi^T(t) w}{k^*}$$

Since $y_m(t) = W_m(p) r$, the tracking error is seen to be related to the parameter error by

$$e(t) = W_m(p) \cdot \frac{\phi^T(t) w(t)}{k^*}$$

• According to lemma 8.1. the following adaptation law is chosen

$$\dot{\theta} = -\mathrm{sgn}(k_p) \gamma e(t) w(t)$$

§ 8.4.2. Linear Systems with Higher Relative Degree.

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• The design of adaptive controller for plants with relative degree larger than 1 is both similiar to, and different from, that for plants with relative degree 1.

- Specifically, the choice of control law is quite similar but the choice of adaptation law is very different.

Choice of Control law

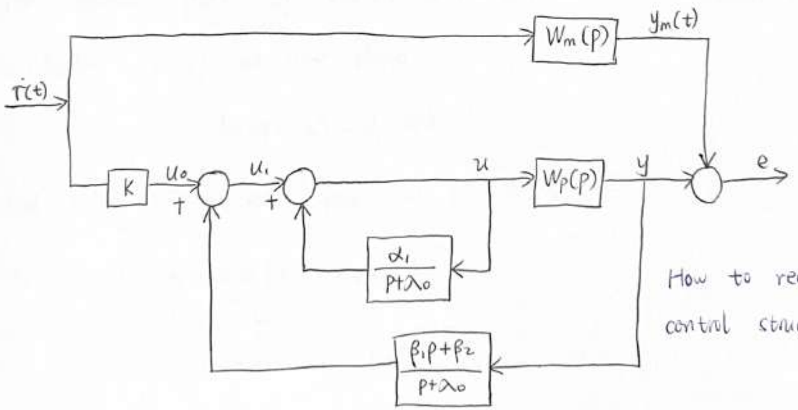
Example: Consider the second order plant described by the transfer function

$$y = \frac{k_p u}{p^2 + a_{p1} p + a_{p2}}.$$

and the reference model

$$y_m = \frac{k_m r}{p^2 + a_{m1} p + a_{m2}}$$

- Let us consider the control structure shown as follows.



How to read this control structure?

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- The closed-loop transfer function from the reference signal r to the plant output y is

$$W_{ry} = k \frac{\frac{p+\lambda_0}{p+\lambda_0+\alpha_1} \frac{k_p}{p^2+a_{p1}p+a_{p2}}}{1+\frac{p+\cancel{\lambda_0}}{p+\lambda_0+\alpha_1} \frac{\beta_1 p+\beta_2}{p\cancel{+\lambda_0}} \frac{k_p}{p^2+a_{p1}p+a_{p2}}} = \frac{k k_p (p+\lambda_0)}{(p+\lambda_0+\alpha_1)(p^2+a_{p1}p+a_{p2}) + k_p(\beta_1 p+\beta_2)}$$

$\Longrightarrow$ Therefore, if the controller parameters $\alpha_1, \beta_1, \beta_2$ and k are chosen such that

$$(p+\lambda_0+\underline{\alpha_1})(p^2+a_{p1}p+a_{p2})+k_p(\underline{\beta_1}p+\underline{\beta_2})=(p+\lambda_0)(p^2+a_{m1}p+a_{m2})$$

$$k=\frac{k_m}{k_p},$$

then $$W_{ry}=\frac{\frac{k_m}{k_p}\cdot \cancel{k_p}\cdot(p+\cancel{\lambda_0})}{(p+\cancel{\lambda_0})(p^2+a_{m1}p+a_{m2})}=\frac{k_m}{p^2+a_{m1}p+a_{m2}}$$

i.e. the closed-loop transfer function W_{ry} becomes identically the same as that of the reference model. Clearly, such choice of parameters exists and is unique

- For general plants of relative degree larger than 1, instead of $\det[pI-\Lambda]=Z_m(p)$, we now chose

$$\det[pI-\Lambda]=Z_m(p)\lambda_1(p)$$

and $\lambda_1(p)$ is a Hurwitz polynomial of degree $(n-1-m)$.

- The closed-loop transfer function is

$$W_{ry}=\frac{kk_pZ_p\lambda_1(p)Z_m(p)}{R_p(p)[\lambda(p)+C(p)]+k_pZ_pD(p)}$$

Now, the question is whether in this general case there exist choice of values for k, θ_0, θ_1 and θ_2 such that the above transfer function becomes exactly the same as $W_m(p)$, or equivalently,

$$R_p(\lambda(p)+C(p))+k_pZ_pD(p)=\lambda_1Z_pR_m(p). \qquad (10)$$

- The answer to this question can be obtained from the following

Lemma: Let $A(p)$ and $B(p)$ be polynomials of degree n_1 and n_2. If $A(p)$ and $B(p)$ are relatively prime. then there exist polynomials $M(p)$ and $N(p)$ such that

$$A(p)M(p) + B(p)N(p) = \boxed{A^*(p)},$$

where $A^*(p)$ is an arbitrary polynomial.

- This Lemma can be used straightforwardly to answer our question regarding (10). $R_p(\lambda(p)+C(p)) + k_pZ_pD(p) = \boxed{\lambda_1 Z_p R_m(p)}$

By regarding R_p as $A(p)$, k_pZ_p as $B(p)$ and $\lambda_1 Z_p R_m$ as $A^*(p)$, we conclude that there exist polynomials $\lambda(p)+C(p)$ and $D(p)$ s.t. (10) is satisfied.

Choice of Adaptation law

- We again use a control law of the form

$$u = \theta^T(t)w(t)$$

Using a similar reasoning as before, we can again obtain the output y in the form of $y(t) = W_m(p)r + W_m(p)[\phi^T(t)w/k^*]$ and the tracking error is in the form of

$$e(t) = W_m(p)[\phi^T w/k^*] \qquad (11)$$

- However, the choice of adaptation law given by $\dot{\theta} = -sgn(k_p)\gamma e(t)w(t)$ cannot be used. (Unlike to the case of plant with relative degree 1) because now the reference model transfer function $W_m(p)$ is no longer SPR.

- A famous technique called <u>error augmentation</u> (增强) can be used to avoid the difficulty in finding an adaptation law for $e(t) = W_m(p)[\phi^T(t) w / k^*]$

- Specifically, let us define an auxiliary (/ɔg'zilyəre/) error $\eta(t)$ by

$$\eta(t) = \theta^T(t) W_m(p)[w] - W_m(p)[\theta^T(t) w(t)] \tag{12}$$

and define an augmented error $\varepsilon(t)$ by combining the tracking error $e(t)$ with the auxiliary error $\eta(t)$ as

$$\varepsilon(t) = e(t) + \alpha(t)\eta(t) \tag{13}$$

~~For~~ Substituting (11), (12) into (13), we obtain

$$\varepsilon(t) = \frac{1}{k^*}\phi^T(t)\bar{w} + \phi_\alpha \eta(t),$$

where $\phi_\alpha = \alpha(t) - \frac{1}{k^*}$, $\bar{w}(t) = W_m(p)[w]$.

- A number of standard techniques to be discussed in §8.7 can be used to demonstrate that the controller parameters $\theta(t)$ and the parameter $\alpha(t)$ for forming the augmented error are updated by

$$\dot{\theta} = -\frac{\operatorname{sgn}(k_p)\gamma\varepsilon\bar{w}}{1+\bar{w}^T\bar{w}}, \qquad \dot{\alpha} = -\frac{\gamma\varepsilon\eta}{1+\bar{w}^T\bar{w}} \tag{14}$$

- With the control law $u = \theta^T(t) w(t)$ and adaptation law (14), global convergence of the tracking error can be shown.

§ 8.5. ~~Apa~~ Adaptive Control of Nonlinear Systems.

Problem Statement We consider n^{th}-order nonlinear systems

$$y^{(n)} + \sum_{i=1}^{n} \alpha_i f_i(x,t) = bu, \tag{15}$$

where $x = [y, \dot{y}, \cdots, y^{(n-1)}]^T$ is the state vector

f_i are known nonlinear functions of the state and time and the parameters α_i and b are unknown constants. We assume that the state is measured, and that the sign of b is known. One <u>example</u> of such dynamics is

$$m\ddot{x} + c f_1(\dot{x}) + k f_2(x) = u,$$

which represents a mass-spring-damper system with nonlinear friction and nonlinear damping.

- The <u>objective</u> of the adaptive control design to make the output asymptotically tracks a desired output $y_d(t)$ despite the parameter uncertainty

- Let us rewrite (15) as

$$h y^{(n)} + \sum_{i=1}^{n} a_i f_i(x,t) = u. \qquad (16)$$

by dividing both sides by the unknown constant b, where $h = \frac{1}{b}$ and $a_i = \frac{\alpha_i}{b}$

Choice of control law

- define a combined error

$$s = e^{(n-1)} + \lambda_{n-2} e^{(n-2)} + \cdots + \lambda_0 e$$

Note that s can be rewritten as

$$s = y^{(n-1)} - y_r^{(n-1)},$$

$$\downarrow$$

$$y_d^{(n-1)} - \lambda_{n-2} e^{(n-2)} - \cdots - \lambda_0 e.$$

- Consider the control law

$$u = h y_r^{(n)} - ks + \sum_{i=1}^{n} a_i f_i(x,t), \qquad (17)$$

where k is a constant of the same sign as h,

$$y_r^{(n)} = y_d^{(n)} - \lambda_{n-2} e^{(n-1)} - \cdots - \lambda_0 \dot{e}$$

- Substituting (17) into (16), we see

$$h y^{(n)} + \sum_{i=1}^{n} a_i \cancel{f_i(x,t)} = h y_r^{(n)} - ks + \sum_{i=1}^{n} a_i \cancel{f_i(x,t)}$$

$$h\dot{s} + ks = 0$$

and therefore gives exponential convergence of s, which, in turn, guarantees the convergence of e.

Choice of adaptation law

- For our adaptive control, the control law (17) is replaced by

$$u = \hat{h} y_r^{(n)} - ks + \sum_{i=1}^{n} \hat{a}_i f_i(x,t),$$

where h and the a_i have been replaced by their estimated values. The tracking error from this control law can be easily shown to be

$$h\dot{s} + ks = \tilde{h} y_r^{(n)} + \sum_{i=1}^{n} \tilde{a}_i f_i(x,t).$$

- This can be rewritten as

$$s = \frac{1/h}{p + (k/h)} \left[\tilde{h} y_r^{(n)} + \sum_{i=1}^{n} \tilde{a}_i f_i(x,t) \right].$$

Lemma 8.1. suggests us to choose the following adaptation law

$$\dot{\hat{h}} = -\gamma \, \mathrm{sgn}(h) \, s y_r^{(n)}$$

$$\dot{\hat{a}}_i = -\gamma \, \mathrm{sgn}(h) \, s f_i$$