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Chapter 7. Reliable Control for Stochastic Switched System with state delay

7.1. Problem formulation

$$(7.1)\quad \begin{cases} d\,x(t) = (A_{k(t)}\,x + B_{k(t)}\,u + f_{k(t)}\,(x(t)))dt + g_{k(t)}\,(x_t)\,dW(t) \\ X_{t_0} = \phi(s), \quad s \in [-r, 0]. \end{cases}$$

- $t \geq t_0$. $t_0 \in \mathbb{R}_+$.
- $x(t) \in \mathbb{R}^n$ is the system state vector, $u(t) \in \mathbb{R}^q$: control system input.
- $k: [t_0, \infty) \to S$. $S = \{1, 2, \cdots N\}$ piecewise constant ftn., switching signal

To guarantee that (7.1) has a unique regular solution.
assume that i) $f_i \in L_{ad}(\Omega, L[a,b])$.
$g_i \in L_{ad}(\Omega, L^2[a,b])$,

f_i. g_i satisfy Lipschitz condition.

To analyse the reliable stabilization w.r.t. actuator failures,
$i \in S$. consider the decomposition $B^i = B^i_\Sigma + B^i_{\bar{\Sigma}}$.

Σ: the set of actuator, susceptible to failure

$\bar{\Sigma}$: $\{1, 2, \cdots q\} - \Sigma$: robust to failures, essential to stabilize the system

Applying the control input u as the form $u(t) = K_i\, x(t)$ (7.2),
(7.1) becomes.
$$d\,x(t) = ((A_i + B_{i\sigma}\,K_i)\,x + f_i\,(x_t))\,dt + g_i\,(x_t)\,dW(t)$$
$$X_{t_0} = \phi(s).$$
$\forall i \in S$, $t \in [t_k, t_{k+1}]$

7.2 Stability Analysis

Thm 7.1

Assume $\varepsilon_i > 0$. P_i : is positive definite matrix.

- $(A_i + B_i k_i + \frac{1}{2}\gamma_i I)^T P_i + P_i (A_i + B_i k_i + \frac{1}{2}\gamma_i I) + \frac{1}{\varepsilon_i} P_i^2 + \varepsilon_i \bar{P}_i \|U_i\|^2 I + \alpha_i P_i =$

- $tr[g_i^T(\psi) P_i g_i(\psi)] \leq 2\gamma_i \bar{q}_i \psi_{(0)}^T P_i \psi(0)$, $\left(\begin{array}{l} \alpha_i > 0 \quad \bar{q}_i > 1 \quad \gamma_i > 0 \\ \|f_i(\psi)\|^2 \leq \bar{q}_i \|U_i\|^2 \|\psi_{(0)}\|^2 \end{array} \right.$

- dwell-time-type condition

$$t_k - t_{k-1} \geq \frac{1}{\alpha_i} \ln \left(\frac{a_i d_{i-1} e^{r d_i}}{b_i d_i} \right) > 0$$

$a_i = \lambda_{max}(P_i)$, $b_i = \lambda_{min}(P_i)$,

$d_i < d_{i-1} < 1$, s.t. $\lim_{i \to \infty} d_{i-1} = 0$.

Then, switched system (7.3) is asymptotically stable in the m.s.

Pf $dx(t) = (A_i + B_i k_i) x_t + f_i(x_t) dt + g_i(x_t) dW(t)$,

Define $V_i(x) = x^T P_i x$. $(V_i)_t = X^T P \dot{X} + \dot{X}^T P X$ & $(V_i)_x = x^T P_i I + I P_i x$

$(V_i)_{xx} = 2 I P_i I$.

According to Itô's formula

$$\mathcal{L} V_i = V_t + V_x \cdot f(t, x(t), u(t)) + \frac{1}{2} tr[g^T(t, x, u) V_{xx} g(t, x, u)],$$

we have

$$\mathcal{L} V_i(x) = 2[((A_i + B_i k_i) x + f_i(x_t))^T P_i x + x^T P_i ((A_i + B_i k_i) x + f_i(x_t))$$
$$+ tr[g_i^T(x_t) P_i g_i(x_t)],$$

By the condition $tr[g_i^T(\psi) P_i g_i(\psi)] \leq 2\gamma_i \bar{q}_i \psi_{(0)}^T P_i \psi(0)$,

$$\Rightarrow \mathcal{L} V_i \leq 2[x^T (A_i^T P_i + P_i A_i + 2k_i^T B_i^T P_i) x + f_i^T(x_t) P_i x + x^T P_i f_i(x_t)$$
$$+ \gamma_i \bar{q}_i x^T x]$$

From reference 1. we also have

$$2x^T P_i f_i(x_t) \le x^T(\varepsilon_i \bar{q}_i \|U_i\|^2 I + \frac{1}{\varepsilon_i} P^2) x$$

$$\mathcal{L} V_i(x) \le 2x^T\left((A_i + B_i k_i)^T P_i + P_i(A_i + B_i k_i) + \varepsilon_i \bar{q}_i \|U_i\|^2 I + \frac{1}{\varepsilon_i} P_i^2 + \gamma_i \bar{q}_i I\right) x$$

$$\underline{\text{Riccati-like matrix}} - \alpha_i x^T P_i x = -\alpha_i V_i(x) < 0$$

Ito' Lemma,

$$dV_i(x) = \mathcal{L} V_i(x) dt + \frac{\partial V_i}{\partial x} g_i(x_t) dW(t)$$, taking expectation,

we have

$$E[dV_i(x)] = E[\mathcal{L} V_i(x)\, dt + \frac{\partial V_i}{\partial x} g_i(x_t) dW_t]$$. since $E[dW_t] = 0$

$$E[dV_i(x)] = E[\mathcal{L} V_i(x)] dt$$, we know that $\mathcal{L} V_i(x) \le -\alpha_i V_i(x)$

then $E[dV_i(x)] \le -\alpha_i E[V_i(x)] dt$.

Let $m_i(t) = E[V_i(x(t))]$, then $\frac{dm_i(t)}{dt} \le -\alpha_i m_i(t)$

we use $D^+ m_i(t)$ represent the upper derivative (Dini derivative), then we have

$$D^+ m_i(t) \le -\alpha_i m_i(t),$$ (consider first order ODE. $\frac{dm_i}{dt} = -\alpha m_i(t)$
$m_i(t) = m_i(t_0) e^{-\alpha_i(t-t_0)}$

By comparison theorem, $m_i(t)$ satisfies $m_i(t) \le m_i(t_0) e^{-\alpha_i(t-t_0)}$

$$\therefore ① E[x^T(t) P_i x(t)] \le E[x^T(t_0) P_i x(t_0)] e^{-\alpha_i(t-t_0)}$$

Note that
$a_i = \lambda_{max}(P_i)$. $b_i = \lambda_{min}(P_i)$ $\Rightarrow$ $b_i x^T(t) x(t) \le x^T(t) P_i x(t) \le a_i x^T(t) x(t)$

$$② b_i E[\|x(t)\|^2] \le E[x^T(t) P_i x(t)] \cancel{e^{-\alpha_i(t-t_0)}} \le a_i E[\|x(t)\|^2]$$

Combine ① and ②. we obtain

$$E[\|x(t)\|^2] \le \frac{a_i}{b_i} E[\|x(t_0)\|^2] e^{-\alpha_i(t-t_0)}$$

In the interval $[t_k, t_{k+1})$, $t_0 = t_k$. So

$$E[\|x(t)\|^2] \le \frac{a_i}{b_i} E[\|x_{t_k}\|^2] e^{-\alpha_i (t-t_k)}$$

From the dwell time condition, $(t_k - t_{k-1} \ge \frac{1}{\alpha_i} \ln(\frac{a_i d_{i-1} e^{r\alpha_i}}{b_i d_i})$, we obtain

$$e^{-\alpha_i(t_{k+1}-t_k)} \le \frac{b_i d_i}{a_i d_{i-1} e^{r\alpha_i}}$$

$$E[\|x(t_{k+1})\|^2] \le \frac{a_i}{b_i} E[\|x_{t_k}\|^2] \frac{b_i d_i}{a_i d_i e^{r\alpha_i}} E[\|x_{t_k}\|^2] \le d_i E[\|x_{t_k}\|^2]$$

$\Rightarrow$ $E[\|x(t_n)\|^2] \le d_n E[\|x_{t_{n-1}}\|^2] \le d_n d_{n-1} \cdots \le d_n d_{n-1} \cdots d_1 E[\|x_{t_0}\|^2]$

Therefore, we have $\lim_{n \to \infty} E[\|x(t)\|^2] = 0$, since $\lim_{n\to\infty} d_i = 0$

Theorem 2.2.

- Riccati-like matrix eqn.

$$(A_i + \frac{1}{2}\gamma_i I)^T P_i + P_i (A_i + \frac{1}{2}\gamma_i I) + P_i(\frac{1}{\varepsilon_i} I - \varepsilon_i^* B_{\bar{\sigma}} B_{\bar{\sigma}}^T) P_i$$
$$+ \varepsilon_i \bar{q}_i \|\omega_i\|^2 I + \alpha_i P_i = 0$$

- Other conditions are same as in Theorem 2.1.

$\Rightarrow$ System (2.3) are m.s. globally exponentially stabilized by the control law $u = K_i x$, control gain $K_i = -\frac{1}{2}\varepsilon_i^* B_{i\bar{\sigma}}^T P_i$ for any nonlinear uncertainties and actuator failures.

Moreover, (2.3) is globally asymptotically stable in the m.s.

Chapter 8. Robust Reliable Control for Impulsive Large-Scale Systems

8.1 Problem formulation

Isolated subsystems

$$\begin{cases} \dot{w}^i = (A^i + \Delta A^i) w^i + B^i u^i + f_i(w^i) & t \neq t_k \\ \Delta w^i(t) = C_{ik} w^i(t^-) & t = t_k \\ w^i(t_0) = w_0^i \end{cases} \qquad (8.3)$$

Closed-loop system

$$\begin{cases} \dot{w}^i = (A^i + \Delta A^i + B^i K^i) w^i + f_i(w^i) & t \neq t_k \\ \Delta w^i(t) = C_{ik} w^i(t^-) & t = t_k \\ w^i(t_0) = w^i_0 \end{cases} \qquad (8.4)$$

We consider decomposition $B^i = B^i_{\sigma} + B^i_{\bar{\sigma}}$, then closed-loop system for the faulty case:

$$\begin{cases} \dot{w}^i = (A^i + \Delta A^i + B^i_{\bar{\sigma}} K^i) w^i + f_i(w^i) & t \neq t_k \\ \Delta w^i(t) = C_{ik} w^i(t^-) & t = t_k \\ w^i(t_0) = w_0^i \end{cases}$$

Def: The trivial solution $x \equiv 0$ of the interconnected system is said to be robustly globally exponentially stable if

$\exists\ \lambda, \bar{\lambda} > 0$ s.t.

$$\|x\| \le \bar{\lambda} \|x_0\| e^{-\lambda(t-t_0)} \quad \forall t \ge t_0$$

Assumption A: For $i = 1, 2, \ldots, l$, the admissible parameter uncertainties are defined by

$$\Delta A^i(t) = D^i U^i(t) H^i, \quad D^i, H^i: \text{real matrices}$$

$$\|U^i(t)\| \le 1$$

Lemma: For $\epsilon_1, \xi_1 > 0$ positive definite matrix P.

(i) $2x^T P(\Delta A)x \le x^T(\epsilon_1 P D D^T P + \frac{1}{\epsilon_1} H^T H)x$ &

(ii) $2x^T P f(x) \le x^T(\xi_1 P^2 + \frac{\delta I}{\xi_1})x$ s.t. $\|f(x)\|^2 \le \delta \|x\|^2$, $\delta > 0$

Proof of Thm 8.1 :

Let $w^i(t) = w^i(t; t_0, w_0^i)$ be a solution of system (8.4) and Lyapunov function $V^i(w^i) = w^{iT} P^i w^i$.

$$\dot{V}^i(w^i) = \dot{w}^{iT} P_i w^i + w^{iT} P_i \dot{w}^i$$

$$= w^{iT}\left((A^i + B^i K^i)^T P^i + P^i(A^i + B^i K^i)\right) w^i \underbrace{+ 2w^{iT} P^i(\Delta A^i) w^i}_{\text{By Lemma 8.1 (1)}} + 2w^{iT} P^i \overbrace{f_i(w^i)}^{\|f_i(w^i)\|^2 \le \delta_i \|w^i\|^2}$$

$$\le w^{iT}\Big((A^i + B^i K^i)^T P^i + P^i(A^i + B^i K^i) + \epsilon_{ii} P^i D^i D^{iT} P^i + \frac{1}{\epsilon_{ii}} H^{iT} H^i$$

$$+ \xi_{1i} P^{i2} + \frac{\delta_i I}{\xi_{1i}}\Big) w^i$$

$$= w^{iT}(\sigma P_i) w^i = \sigma_i V^i(w^i),$$

Then, for all $t \in (t_{k-1}, t_k]$, we have

$$V^i(w^i(t)) \le \underline{V^i(w^i(t_{k-1}^+))}\, e^{+\sigma_i(t_k - t_{k-1})} \qquad ①$$

$$V^i(w^i(t_{k-1}^+)) = w^{iT}(t_k^+) P^i w^i(t_k^+)$$

$$= (w^{iT}(t_k^-) + \Delta w^{iT}(t_k)) P^i (w^i(t_k^-) + \Delta w^i(t_k))$$

$$= (w^{iT}(t_k^-) + w^{iT}(t_k^-) C_{ik}^T) P_i (w^i(t_k^-) + C_{ik} w^i(t_k^-))$$

$$= w^{iT}(t_k^-)(I + C_{ik})^T P^i (I + C_{ik}) w^i(t_k^-)$$

$$\le \lambda_{max}(L_{ik}) w^{iT}(t_k) w^i(t_k) \qquad ②$$

$$\le d_{ik} V^i(w^i(t_k^-)), \quad \text{where } d_{ik} = \frac{\lambda_{max}(L_{ik})}{\lambda_{min}(P^i)}, \quad L_{ik} = [I + C_{ik}]^T P^i [I + C_{ik}]$$

By ① & ② we have ($t \in [t_0, t_1]$)

$$V^i(w^i(t)) \le V^i(w_0^i) e^{\sigma_i(t - t_0)}, \quad \text{and for } t \in [t_1, t_2]$$

$$V^i(w^i(t_1^+)) \le d_{i1} V^i(w_0^i) e^{\sigma_i(t - t_0)}, \quad V^i(w^i(t)) \le V^i(w^i(t_1^+)) e^{\sigma_i(t - t_1)}$$

$$\Rightarrow V^i(w^i(t)) \le d_{i1} V^i(w_0^i) e^{\sigma_i(t - t_0)}, \quad \text{for } t \in [t_0, t_2]$$

Generally, for $t \in [t_{k-1}, t_k]$, we have

$$V^i(w^i(t)) \le V^i(w_0^i) e^{(\sigma_i + \nu_i)(t - t_0)}$$

since $(\ln d_{ik} - \nu_i(t_k - t_{k-1}) \le 0)$

$t \ge t_0, \quad 0 < \nu_i < -\sigma_i$

Moreover,

$$\lambda_{min}(P^i)\|w^i\|^2 \le V^i(w^i) \le V^i(w_0^i)e^{(\sigma_i+\nu_i)(t-t_0)}$$
$$= w_0^{i^T} P^i w_0^i e^{(\sigma_i+\nu_i)(t-t_0)}$$
$$\le \lambda_{max}(P^i)\|w_0\|^2 e^{(\sigma_i+\nu_i)(t-t_0)}.$$

which leads to $\|w^i\| \le \gamma_i \|w_0^i\| e^{(\sigma_i+\nu_i)(t-t_0)/2}$ $(t \ge t_0)$, $\gamma_i = \sqrt{\frac{\lambda_{max}(P_i)}{\lambda_{min}(P_i)}}$ ☐

§3. State estimation

Consider ~~ILSS~~ isolated input-output impulsive system:

$$\begin{cases} \dot{w}_i = (A^i + \Delta A^i)w^i + B^i u^i + f_i(w^i) & t \ne t_k, \quad k \in \mathbb{N} \\ \Delta w^i = C_{ik} w^i(t^-). \\ y^i(t) = C^i w^i(t) \\ w^i(t_0) = w_0^i, \end{cases}$$

We also define Luenberger observer by

$$\begin{cases} \dot{\hat{w}}^i = (A^i + \Delta A^i)\hat{w}^i + B^i u^i + f_i(\hat{w}^i) + L^i(y^i - C^i\hat{w}^i) & t \ne t_k \\ \Delta\hat{w}^i(t) = C_{ik}\hat{w}^i(t^-), & t = t_k \\ \hat{w}^i(t_0) = \hat{w}_0^i. \end{cases}$$

Define state estimation error vector by $e^i = w^i - \hat{w}^i$. Then closed-loop error system becomes

$$\begin{cases} \dot{e}^i = (A^i + \Delta A^i - L^i C^i)e^i + f_i(w^i) - f_i(\hat{w}^i). & t \ne t_k \\ \Delta e^i(t) = C_{ik} e^i(t^-) \\ e^i(t_0) = w_0^i - \hat{w}_0^i =: e_0^i. \end{cases}$$

We adopt the same analysis of stability followed in the section §2. to establish the observability problem of isolated input-output impulsive subsystem